

THE NUCLEAR SCISSORS MODE FROM VARIOUS ASPECTS

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Abstract

Three methods to describe collective motion, Random Phase Approximation (RPA), Wigner Function Moments (WFM) and the Green's Function (GF) method are compared in detail and their physical content analyzed on an example of a simple model, the harmonic oscillator with quadrupole-quadrupole residual interaction. It is shown that they give identical formulae for eigenfrequencies and transition probabilities of all collective excitations of the model, including the scissors mode, which is the subject of our special attention. The exact relation between the RPA and WFM variables and the respective dynamical equations is established. The transformation of the RPA spectrum into the one of WFM is explained. The very close connection of the WFM method with the GF one is demonstrated. The normalization factor of the "synthetic" scissors state and its overlap with physical states are calculated analytically. The orthogonality of the spurious state to all physical states is proved rigorously. A differential equation describing the current lines of RPA modes is established and the current lines of the scissors mode analyzed as a superposition of rotational and irrotational components.

1 Introduction

The full analysis of the scissors mode in the framework of a solvable model (harmonic oscillator with quadrupole–quadrupole residual interaction (HO+QQ)) was given in [1]. Several points in the understanding of the nature of this mode were clarified: for example, its coexistence with the isovector giant quadrupole resonance (IVGQR), the decisive role of the Fermi surface deformation, and several things more. The Wigner Function Moments (WFM) method was applied to derive analytical expressions for currents of both coexisting modes, their excitation energies, magnetic and electric transition probabilities. Our formulae for energies turned out to be identical with those derived by Hamamoto and Nazarewicz [2] in the framework of the RPA. However, little details are given in that reference of letter form. In [3] we investigated the relation between formulas for transition probabilities derived by the two methods. It was shown there, that also these formulas are identical. This coincidence motivated us to undertake a systematic comparison of the two approaches and to understand the connection and differences between them. This is the goal of the present paper. One of the important subjects of this comparison are the current distributions. The WFM method, a priori, can not give the exact results, because it deals only with integrals over the whole phase space. It would therefore be very interesting to evaluate the accuracy of this approximation by comparing the results with the currents obtained from RPA. Unfortunately, even for this simple model (HO+QQ) it is impossible to derive in RPA the closed analytical expressions for currents of the scissors mode and IVGQR. That is why we consider in addition the Green’s Function (GF) method, which allows one to find explicit expressions for the currents directly.

The HO+QQ model is a very convenient ground for this kind of investigation, because all results can be obtained analytically. There is no need to describe the merits of the RPA or of the GF method – they are very well known [4]. It is necessary, however, to say a few words about the WFM. Its idea is based on the virial theorems of Chandrasekhar and Lebovitz [5]. Instead of writing the equations of motion for microscopic amplitudes of particle hole excitations (RPA), one writes the dynamical equations for various multipole phase space moments of a nucleus. This allows one to achieve a more direct physical interpretation of the studied phenomenon without going into its detailed microscopic structure. The WFM method was successfully

applied to the study of isoscalar and isovector giant multipole resonances and low-lying collective modes of rotating and nonrotating nuclei with various realistic forces [6]. The results of WFM were always very close to similar results obtained with the help of RPA. In principle, this should be expected, because the basis of both methods is the same: Time Dependent Hartree–Fock (TDHF) theory in its small amplitude approximation. On the other hand it is evident, that they are in general not equivalent, because the truncation scheme is different. The detailed analysis of the interplay of the two methods turns out to be useful also from a “practical” point of view: first and most importantly it allows one to obtain additional insight into the nature of the scissors mode; second we find new exact mathematical results for the considered model.

The paper is organized as follows. In Section 2 we recall the principal points of the WFM formalism and give a summary of the key results of [1] obtained by applying this method to the HO+QQ model. The same model is considered in Section 3 in the frame of RPA: the formulae for eigenfrequencies, electric and magnetic transition probabilities of the scissors mode are derived, the “synthetic” scissors and spurious state are analyzed. The exact interrelation between the RPA and WFM methods and between their variables is established in Section 4. Section 5 is dedicated to the GF method. The three methods are applied to derive analytical formulae for lines of currents in Section 6. The mutual interplay of the three methods is discussed in the conclusion. Various mathematical details are given in Appendices A and B.

2 The WFM method

The basis of the method is the TDHF equation for the one-body density matrix $\rho^\tau(\mathbf{r}_1, \mathbf{r}_2, t) = \langle \mathbf{r}_1 | \hat{\rho}^\tau(t) | \mathbf{r}_2 \rangle$:

$$i\hbar \frac{\partial \hat{\rho}^\tau}{\partial t} = [\hat{H}^\tau, \hat{\rho}^\tau], \quad (1)$$

where \hat{H}^τ is the one-body self-consistent mean field Hamiltonian depending implicitly on the density matrix and τ is an isotopic spin index. It is convenient to modify equation (1) introducing the Wigner transform of the density matrix

$$f^\tau(\mathbf{r}, \mathbf{p}, t) = \int d^3s \exp(-i\mathbf{p} \cdot \mathbf{s}/\hbar) \rho^\tau(\mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{r} - \frac{\mathbf{s}}{2}, t) \quad (2)$$

and of the Hamiltonian

$$H_W^\tau(\mathbf{r}, \mathbf{p}) = \int d^3s \exp(-i\mathbf{p} \cdot \mathbf{s}/\hbar) (\mathbf{r} + \frac{\mathbf{s}}{2} | \hat{H}^\tau | \mathbf{r} - \frac{\mathbf{s}}{2}). \quad (3)$$

Using (2,3) one arrives [4] at

$$\frac{\partial f^\tau}{\partial t} = \frac{2}{\hbar} \sin \left\{ \frac{\hbar}{2} [(\nabla)^H \cdot (\nabla^p)^f - (\nabla^p)^H \cdot (\nabla)^f] \right\} H_W^\tau f^\tau, \quad (4)$$

where the upper index on the bracket stands for the function on which the operator in these brackets acts. It is shown in [6, 7], that by integrating equation (4) over the phase space $\{\mathbf{p}, \mathbf{r}\}$ with the weights $x_{i_1} x_{i_2} \dots x_{i_k} p_{i_{k+1}} \dots p_{i_{n-1}} p_{i_n}$, where k runs from 0 to n , one can obtain a closed finite set of dynamical equations for Cartesian tensors of the rank n . Taking linear combinations of these equations one is able to represent them through irreducible tensors, which play the role of collective variables of the problem. However, it is more convenient to derive the dynamical equations directly for irreducible tensors using the technique of tensor products [8]. For this it is necessary to rewrite the Wigner function equation (4) in terms of cyclic variables

$$\frac{\partial f^\tau}{\partial t} = \frac{2}{\hbar} \sin \left\{ \frac{\hbar}{2} \sum_{\alpha=-1}^1 (-1)^\alpha [(\nabla_{-\alpha})^H \cdot (\nabla_\alpha^p)^f - (\nabla_{-\alpha}^p)^H \cdot (\nabla_\alpha)^f] \right\} H_W^\tau f^\tau, \quad (5)$$

with

$$\begin{aligned} \nabla_{+1} &= -\frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right), & \nabla_0 &= \frac{\partial}{\partial x_3}, & \nabla_{-1} &= \frac{1}{\sqrt{2}} \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right), \\ r_{+1} &= -\frac{1}{\sqrt{2}} (x_1 + i x_2), & r_0 &= x_3, & r_{-1} &= \frac{1}{\sqrt{2}} (x_1 - i x_2) \end{aligned}$$

and the analogous definitions for ∇_{+1}^p , ∇_0^p , ∇_{-1}^p , and p_{+1} , p_0 , p_{-1} . The required equations are obtained by integrating (5) with different tensor products of r_α and p_α . Here we consider the case $n = 2$.

2.1 Model Hamiltonian, Equations of motion

The microscopic Hamiltonian of the model, harmonic oscillator plus separable quadrupole-quadrupole residual interaction is given by

$$\begin{aligned} H &= \sum_{i=1}^A \left(\frac{\hat{\mathbf{p}}_i^2}{2m} + \frac{1}{2} m \omega^2 \mathbf{r}_i^2 \right) + \bar{\kappa} \sum_{\mu=-2}^2 (-1)^\mu \sum_i^Z \sum_j^N q_{2-\mu}(\mathbf{r}_i) q_{2\mu}(\mathbf{r}_j) \\ &+ \frac{1}{2} \kappa \sum_{\mu=-2}^2 (-1)^\mu \left\{ \sum_{i \neq j}^Z q_{2-\mu}(\mathbf{r}_i) q_{2\mu}(\mathbf{r}_j) + \sum_{i \neq j}^N q_{2-\mu}(\mathbf{r}_i) q_{2\mu}(\mathbf{r}_j) \right\}, \end{aligned} \quad (6)$$

where the quadrupole operator $q_{2\mu} = \sqrt{16\pi/5} r^2 Y_{2\mu}$ and N, Z are the numbers of neutrons and protons, respectively. The mean field potential for protons (or neutrons) is

$$V^\tau(\mathbf{r}, t) = \frac{1}{2} m \omega^2 r^2 + \sum_{\mu=-2}^2 (-1)^\mu \tilde{Z}_{2-\mu}^\tau(t) q_{2\mu}(\mathbf{r}), \quad (7)$$

where $\tilde{Z}_{2\mu}^n = \kappa Q_{2\mu}^n + \bar{\kappa} Q_{2\mu}^p$, $\tilde{Z}_{2\mu}^p = \kappa Q_{2\mu}^p + \bar{\kappa} Q_{2\mu}^n$ and the quadrupole moments $Q_{2\mu}^\tau(t)$ are defined as

$$Q_{2\mu}^\tau(t) = \int d\{\mathbf{p}, \mathbf{r}\} q_{2\mu}(\mathbf{r}) f^\tau(\mathbf{r}, \mathbf{p}, t)$$

with $\int d\{\mathbf{p}, \mathbf{r}\} \equiv 2(2\pi\hbar)^{-3} \int d^3p \int d^3r$, where the factor 2 appears due to summation over spin degrees of freedom. To simplify notation we omit spin indices, because we consider spin saturated system without the spin-orbit interaction.

Substituting spherical functions by tensor products $r^2 Y_{2\mu} = \sqrt{\frac{15}{8\pi}} r_{2\mu}^2$, where

$$r_{\lambda\mu}^2 \equiv \{r \otimes r\}_{\lambda\mu} = \sum_{\sigma, \nu} C_{1\sigma, 1\nu}^{\lambda\mu} r_\sigma r_\nu$$

and $C_{1\sigma, 1\nu}^{\lambda\mu}$ is the Clebsch-Gordan coefficient, one has

$$V^\tau = \frac{1}{2} m \omega^2 r^2 + \sum_{\mu} (-1)^\mu Z_{2-\mu}^\tau r_{2\mu}^2. \quad (8)$$

Here

$$Z_{2\mu}^n = \chi R_{2\mu}^n + \bar{\chi} R_{2\mu}^p, \quad Z_{2\mu}^p = \chi R_{2\mu}^p + \bar{\chi} R_{2\mu}^n, \quad \chi = 6\kappa, \quad \bar{\chi} = 6\bar{\kappa},$$

$$R_{\lambda\mu}^\tau(t) = \int d\{\mathbf{p}, \mathbf{r}\} r_{\lambda\mu}^2 f^\tau(\mathbf{r}, \mathbf{p}, t). \quad (9)$$

Integration of equation (5) with the weights $r_{\lambda\mu}^2$, $(rp)_{\lambda\mu} \equiv \{r \otimes p\}_{\lambda\mu}$ and $p_{\lambda\mu}^2$ yields the following set of equations [1]:

$$\begin{aligned} \frac{d}{dt} R_{\lambda\mu}^\tau - \frac{2}{m} L_{\lambda\mu}^\tau &= 0, \quad \lambda = 0, 2 \\ \frac{d}{dt} L_{\lambda\mu}^\tau - \frac{1}{m} P_{\lambda\mu}^\tau + m \omega^2 R_{\lambda\mu}^\tau - 2\sqrt{5} \sum_{j=0}^2 \sqrt{2j+1} \{_{2\lambda 1}^{11j}\} (Z_2^\tau R_j^\tau)_{\lambda\mu} &= 0, \quad \lambda = 0, 1, 2 \\ \frac{d}{dt} P_{\lambda\mu}^\tau + 2m \omega^2 L_{\lambda\mu}^\tau - 4\sqrt{5} \sum_{j=0}^2 \sqrt{2j+1} \{_{2\lambda 1}^{11j}\} (Z_2^\tau L_j^\tau)_{\lambda\mu} &= 0, \quad \lambda = 0, 2 \end{aligned} \quad (10)$$

where $\{_{2\lambda 1}^{11j}\}$ is the Wigner $6j$ -symbol and the following notation is introduced

$$P_{\lambda\mu}^\tau(t) = \int d\{\mathbf{p}, \mathbf{r}\} p_{\lambda\mu}^2 f^\tau(\mathbf{r}, \mathbf{p}, t), \quad L_{\lambda\mu}^\tau(t) = \int d\{\mathbf{p}, \mathbf{r}\} (rp)_{\lambda\mu} f^\tau(\mathbf{r}, \mathbf{p}, t).$$

By definition $q_{2\mu} = \sqrt{6}r_{2\mu}^2$, $Q_{2\mu}^\tau = \sqrt{6}R_{2\mu}^\tau$, $R_{00}^\tau = -Q_{00}^\tau/\sqrt{3}$ with $Q_{00}^\tau = N_\tau \langle r^2 \rangle$ being the mean square radius of neutrons or protons. The tensor $L_{1\nu}^\tau$ is connected with angular momentum by the relations $L_{10}^\tau = \frac{i}{\sqrt{2}}I_3^\tau$, $L_{1\pm 1}^\tau = \frac{1}{2}(I_2^\tau \mp iI_1^\tau)$.

We rewrite equations (10) in terms of the isoscalar and isovector variables $R_{\lambda\mu} = R_{\lambda\mu}^n + R_{\lambda\mu}^p$, $\bar{R}_{\lambda\mu} = R_{\lambda\mu}^n - R_{\lambda\mu}^p$ (and so on) with the isoscalar $\kappa_0 = (\kappa + \bar{\kappa})/2$ and isovector $\kappa_1 = (\kappa - \bar{\kappa})/2$ strength constants. There is no problem to solve these equations numerically. However, we want to simplify the situation as much as possible to get the results in analytical form giving us a maximum of insight into the nature of the modes.

1) We consider the problem in small-amplitude approximation. Writing all variables as a sum of their equilibrium value plus a small deviation

$$\begin{aligned} R_{\lambda\mu}(t) &= R_{\lambda\mu}^{eq} + \mathcal{R}_{\lambda\mu}(t), & P_{\lambda\mu}(t) &= P_{\lambda\mu}^{eq} + \mathcal{P}_{\lambda\mu}(t), & L_{\lambda\mu}(t) &= L_{\lambda\mu}^{eq} + \mathcal{L}_{\lambda\mu}(t), \\ \bar{R}_{\lambda\mu}(t) &= \bar{R}_{\lambda\mu}^{eq} + \bar{\mathcal{R}}_{\lambda\mu}(t), & \bar{P}_{\lambda\mu}(t) &= \bar{P}_{\lambda\mu}^{eq} + \bar{\mathcal{P}}_{\lambda\mu}(t), & \bar{L}_{\lambda\mu}(t) &= \bar{L}_{\lambda\mu}^{eq} + \bar{\mathcal{L}}_{\lambda\mu}(t), \end{aligned}$$

we linearize the equations of motion in $\mathcal{R}_{\lambda\mu}$, $\mathcal{P}_{\lambda\mu}$, $\mathcal{L}_{\lambda\mu}$ and $\bar{\mathcal{R}}_{\lambda\mu}$, $\bar{\mathcal{P}}_{\lambda\mu}$, $\bar{\mathcal{L}}_{\lambda\mu}$.

2) We study non-rotating nuclei, i.e. nuclei with $L_{1\nu}^{eq} = \bar{L}_{1\nu}^{eq} = 0$.

3) Only axially symmetric nuclei with $R_{2\pm 2}^{eq} = R_{2\pm 1}^{eq} = \bar{R}_{2\pm 2}^{eq} = \bar{R}_{2\pm 1}^{eq} = 0$ are considered.

4) Finally, we take

$$\bar{R}_{20}^{eq} = \bar{R}_{00}^{eq} = 0. \quad (11)$$

This means that equilibrium deformation and mean square radius of neutrons are supposed to be equal to that of protons.

Due to the approximation (11) the equations for isoscalar and isovector systems are decoupled. Further, due to the axial symmetry the angular momentum projection is a good quantum number. As a result, every set of equations splits into five independent subsets with $\mu = 0, \pm 1, \pm 2$. The detailed derivation of formulae for eigenfrequencies and transition probabilities together with all necessary explanations are given in [1]. Here we write out only the final results required for the comparison with respective results obtained in the framework of RPA.

2.2 Isoscalar eigenfrequencies

The isoscalar subset of equations with $\mu = 1$ is

$$\begin{aligned}\dot{\mathcal{R}}_{21} - 2\mathcal{L}_{21}/m &= 0, \\ \dot{\mathcal{L}}_{21} - \mathcal{P}_{21}/m + [m\omega^2 + 2\kappa_0(Q_{20}^{eq} + 2Q_{00}^{eq})] \mathcal{R}_{21} &= 0, \\ \dot{\mathcal{P}}_{21} + 2[m\omega^2 + \kappa_0 Q_{20}^{eq}] \mathcal{L}_{21} - 6\kappa_0 Q_{20}^{eq} \mathcal{L}_{11} &= 0, \\ \dot{\mathcal{L}}_{11} &= 0.\end{aligned}\tag{12}$$

Using the self-consistent value of the strength constant $\kappa_0 = -\frac{m\bar{\omega}^2}{4Q_{00}}$, (see Appendix A) and the standard definition of the deformation parameter $Q_{20} = Q_{00}\frac{4}{3}\delta$ we reduce (12) to

$$\begin{aligned}\dot{\mathcal{R}}_{21} - 2\mathcal{L}_{21}/m &= 0, \\ \dot{\mathcal{L}}_{21} - \mathcal{P}_{21}/m &= 0, \\ \dot{\mathcal{P}}_{21} + 2m\bar{\omega}^2[(1 + \frac{\delta}{3})\mathcal{L}_{21} + \delta\mathcal{L}_{11}] &= 0, \\ \dot{\mathcal{L}}_{11} &= 0.\end{aligned}\tag{13}$$

Imposing the time evolution via $e^{-i\Omega t}$ for all variables one transforms (13) into a set of algebraic equations. The eigenfrequencies are found from its characteristic equation which reads

$$\Omega^2[\Omega^2 - 2\bar{\omega}^2(1 + \delta/3)] = 0.\tag{14}$$

The nontrivial solution of this equation gives the frequency of the $\mu = 1$ branch of the isoscalar GQR

$$\Omega^2 = \Omega_{is}^2 = 2\bar{\omega}^2(1 + \delta/3).\tag{15}$$

Taking into account the relation (A.7) we find that this result coincides with that of [9]. The trivial solution $\Omega = \Omega_0 = 0$ is characteristic of nonvibrational mode corresponding to the obvious integral of motion $\mathcal{L}_{11} = \text{const}$ responsible for the rotational degree of freedom. This is usually called the ‘spurious’ or ‘Goldstone’ mode. It is easy to find from (13) that $\mathcal{L}_{11} = 0$.

2.3 Isovector eigenfrequencies

The information about the scissors mode is contained in the subset of isovector equations with $\mu = 1$

$$\begin{aligned}
\dot{\bar{\mathcal{R}}}_{21} - 2\bar{\mathcal{L}}_{21}/m &= 0, \\
\dot{\bar{\mathcal{L}}}_{21} - \bar{\mathcal{P}}_{21}/m + [m\omega^2 + \kappa Q_{20}^{eq} + 4\kappa_1 Q_{00}^{eq}] \bar{\mathcal{R}}_{21} &= 0, \\
\dot{\bar{\mathcal{P}}}_{21} + 2[m\omega^2 + \kappa_0 Q_{20}^{eq}] \bar{\mathcal{L}}_{21} - 6\kappa_0 Q_{20}^{eq} \bar{\mathcal{L}}_{11} &= 0, \\
\dot{\bar{\mathcal{L}}}_{11} + 3\kappa_0 Q_{20}^{eq} \bar{\mathcal{R}}_{21} &= 0.
\end{aligned} \tag{16}$$

Supposing, as usual, the isovector constant κ_1 to be proportional to the isoscalar one, $\kappa_1 = \alpha\kappa_0$, taking the self-consistent value for κ_0 and the standard definition of δ , we reduce (16) to

$$\begin{aligned}
\dot{\bar{\mathcal{R}}}_{21} - 2\bar{\mathcal{L}}_{21}/m &= 0, \\
\dot{\bar{\mathcal{L}}}_{21} - \bar{\mathcal{P}}_{21}/m + m\bar{\omega}^2(1 - \alpha)(1 + \frac{\delta}{3}) \bar{\mathcal{R}}_{21} &= 0, \\
\dot{\bar{\mathcal{P}}}_{21} + 2m\bar{\omega}^2[(1 + \frac{\delta}{3}) \bar{\mathcal{L}}_{21} + \delta \bar{\mathcal{L}}_{11}] &= 0, \\
\dot{\bar{\mathcal{L}}}_{11} - m\bar{\omega}^2\delta(1 - \alpha) \bar{\mathcal{R}}_{21} &= 0.
\end{aligned} \tag{17}$$

Imposing the time evolution via $e^{-i\Omega t}$ one transforms (17) into a set of algebraic equations with the characteristic equation

$$\Omega^4 - 2\Omega^2\bar{\omega}^2(2 - \alpha)(1 + \delta/3) + 4\bar{\omega}^4(1 - \alpha)\delta^2 = 0. \tag{18}$$

Its solutions are

$$\Omega_{\pm}^2 = \bar{\omega}^2(2 - \alpha)(1 + \delta/3) \pm \sqrt{\bar{\omega}^4(2 - \alpha)^2(1 + \delta/3)^2 - 4\bar{\omega}^4(1 - \alpha)\delta^2}. \tag{19}$$

The high-lying solution Ω_+ gives the frequency Ω_{iv} of the $\mu = 1$ branch of the isovector GQR. The low-lying solution Ω_- gives the frequency Ω_{sc} of the scissors mode.

We adjust α from the fact that the IVGQR is experimentally known to lie practically at twice the energy of the isoscalar GQR. In our model the experimental situation is satisfied by $\alpha = -2$. Then

$$\Omega_{iv}^2 = 4\bar{\omega}^2 \left(1 + \frac{\delta}{3} + \sqrt{(1 + \frac{\delta}{3})^2 - \frac{3}{4}\delta^2} \right), \quad \Omega_{sc}^2 = 4\bar{\omega}^2 \left(1 + \frac{\delta}{3} - \sqrt{(1 + \frac{\delta}{3})^2 - \frac{3}{4}\delta^2} \right). \tag{20}$$

The scissors mode energy in the limit of small deformation is

$$E_{sc} \approx \sqrt{\frac{3}{2}} \hbar \omega \delta, \quad (21)$$

which is quite close to the result of Hilton [10]: $E_{sc} \approx \sqrt{1 + 0.66} \hbar \omega \delta$. Taking $\hbar \omega = 45.2/A^{1/3}$ MeV (what corresponds to $r_0 = 1.15$ fm used in [11]), one obtains

$$E_{sc} \approx 55.4 \delta A^{-1/3} \text{ MeV},$$

which practically coincides with the result of Lipparini and Stringari [11] (formula (18): $E_{sc} \simeq 56 \delta A^{-1/3}$ MeV) obtained with the help of a microscopic approach based on the evaluation of sum rules. Both results are not very far from the experimental [12] value: $E_{sc} \approx 66 \delta A^{-1/3}$ MeV.

It is interesting to study the role of the Fermi surface deformation for the formation of IVGQR and the scissors mode. Neglecting the variable $\bar{\mathcal{P}}_{21}(t)$ in (17) we find that the frequency of IVGQR (being determined mainly by the neutron-proton interaction) is changed not very much:

$$\Omega_{iv}^2 = 2\bar{\omega}^2(1 - \alpha)(1 + \delta/3).$$

Comparing this formula (for $\alpha = -2$) with (20) one sees, that in the limit of small deformation one obtains $\Omega_{iv}^2 \simeq 6\omega_0^2$ instead of $\Omega_{iv}^2 \simeq 8\omega_0^2$. One should recall that also for the Isovector Giant Dipole Resonance the distortion of Fermi sphere plays only a minor role.

It is also easy to see that omitting $\bar{\mathcal{P}}_{21}(t)$ in (17), one obtains zero energy for the scissors mode independent of the strength of the residual interaction. Thus, the nuclear elasticity discovered by G.F.Bertsch [13] is the single origin for the restoring force of the scissors mode. So one can conclude that this mode is in its essence a pure quantum mechanical phenomenon. This agrees with the conclusion of the papers [14, 15]: classically (i.e., without Fermi surface deformation) the scissors mode is a zero energy mode.

2.4 Linear response and transition probabilities

A direct way of calculating the reduced transition probabilities is provided by the theory of the linear response of a system to a weak external field

$$\hat{F}(t) = \hat{F} \exp(-i\Omega t) + \hat{F}^\dagger \exp(i\Omega t),$$

where $\hat{F} = \sum_{s=1}^A \hat{f}_s$ is a one-body operator. A convenient form of the response theory is e.g. given by Lane [17] (see also section 4). The matrix elements of the operator \hat{F} obey the relation

$$| \langle \nu | \hat{F} | 0 \rangle |^2 = \hbar \lim_{\Omega \rightarrow \Omega_\nu} (\Omega - \Omega_\nu) \overline{\langle \psi | \hat{F} | \psi \rangle \exp(-i\Omega t)}, \quad (22)$$

where $|0\rangle$ and $|\nu\rangle$ are the stationary wave functions of the ground and unperturbed excited states; ψ is the perturbed wavefunction of the ground state, $\Omega_\nu = (E_\nu - E_0)/\hbar$ are the normal frequencies, the bar means averaging over a time interval much larger than $1/\Omega$, Ω being the frequency of the external field $\hat{F}(t)$.

Electric excitations are described by the operator

$$\hat{F} = \hat{F}_{2\mu}^p = \sum_{s=1}^Z \hat{f}_{2\mu}(s), \quad \hat{f}_{2\mu} = e r^2 Y_{2\mu} = \beta r_{2\mu}^2, \quad \beta = e \sqrt{\frac{15}{8\pi}} \quad (23)$$

whose expectation value (in accordance with [1]) is

$$\langle \psi | \hat{F}_{2\mu}^p | \psi \rangle = \beta R_{2\mu}^p = \frac{1}{2} \beta (R_{2\mu} - \bar{R}_{2\mu}).$$

Transition probabilities are given [1] by the following formulae

$$B(E2)_{sc} = 2 | \langle sc | \hat{F}_{21}^p | 0 \rangle |^2 = \frac{e^2 \hbar}{m} \frac{5}{8\pi} Q_{00} \frac{(1 + \delta/3) \Omega_{sc}^2 - 2(\bar{\omega}\delta)^2}{\Omega_{sc}(\Omega_{sc}^2 - \Omega_{iv}^2)}. \quad (24)$$

$$B(E2)_{iv} = 2 | \langle iv | \hat{F}_{21}^p | 0 \rangle |^2 = \frac{e^2 \hbar}{m} \frac{5}{8\pi} Q_{00} \frac{(1 + \delta/3) \Omega_{iv}^2 - 2(\bar{\omega}\delta)^2}{\Omega_{iv}(\Omega_{iv}^2 - \Omega_{sc}^2)}. \quad (25)$$

$$B(E2)_{is} = 2 | \langle is | \hat{F}_{21}^p | 0 \rangle |^2 = \frac{e^2 \hbar}{m} \frac{5}{8\pi} Q_{00} [(1 + \delta/3) \Omega_{is}^2 - 2(\bar{\omega}\delta)^2] / [\Omega_{is}]^3. \quad (26)$$

These three formulae can be joined into one expression by a simple transformation of the denominators. Really, we have from (19)

$$\begin{aligned} \pm(\Omega_{iv}^2 - \Omega_{sc}^2) &= \pm(\Omega_+^2 - \Omega_-^2) = \pm 2 \sqrt{\bar{\omega}^4 (2 - \alpha)^2 (1 + \delta/3)^2 - 4 \bar{\omega}^4 (1 - \alpha) \delta^2} \\ &= 2 \Omega_\pm^2 - 2 \bar{\omega}^2 (2 - \alpha) (1 + \delta/3) = 2 \Omega_\pm^2 - (2 - \alpha) (\omega_x^2 + \omega_z^2). \end{aligned} \quad (27)$$

Using these relations in formulae (24) and (25) we obtain the expression for the $B(E2)$ values valid for all three excitations

$$B(E2)_\nu = 2 | \langle \nu | \hat{F}_{21}^p | 0 \rangle |^2 = \frac{e^2 \hbar}{m} \frac{5}{16\pi} Q_{00} \frac{(1 + \delta/3) \Omega_\nu^2 - 2(\bar{\omega}\delta)^2}{\Omega_\nu [\Omega_\nu^2 - \bar{\omega}^2 (2 - \alpha) (1 + \delta/3)]}. \quad (28)$$

The isoscalar value (26) is obtained by assuming $\alpha = 1$.

Magnetic excitations are described by the operator

$$\hat{F} = \hat{F}_{1\mu}^p = \sum_{s=1}^Z \hat{f}_{1\mu}(s), \quad \hat{f}_{1\mu} = -i\nabla(rY_{1\mu}) \cdot [\mathbf{r} \times \nabla] \mu_N = \gamma(r\hat{p})_{1\mu}, \quad \mu_N = \frac{e\hbar}{2mc}. \quad (29)$$

Its expectation value was calculated in [1] to be

$$\langle \psi | \hat{F}_{1\mu}^p | \psi \rangle = \gamma L_{1\mu}^p = \frac{\gamma}{2} (L_{1\mu} - \bar{L}_{1\mu}) = \frac{\gamma}{2} (\mathcal{L}_{1\mu} - \bar{\mathcal{L}}_{1\mu}), \quad \gamma = -i \frac{e}{2mc} \sqrt{\frac{3}{2\pi}}.$$

Then one finds the following expressions [1] for transition probabilities

$$B(M1)_{sc} = 2 | \langle sc | \hat{F}_{11}^p | 0 \rangle |^2 = \frac{1 - \alpha}{4\pi} \frac{m\bar{\omega}^2}{\hbar} Q_{00} \delta^2 \frac{\Omega_{sc}^2 - 2(1 + \delta/3)\bar{\omega}^2}{\Omega_{sc}(\Omega_{sc}^2 - \Omega_{iv}^2)} \mu_N^2, \quad (30)$$

$$B(M1)_{iv} = 2 | \langle iv | \hat{F}_{11}^p | 0 \rangle |^2 = \frac{1 - \alpha}{4\pi} \frac{m\bar{\omega}^2}{\hbar} Q_{00} \delta^2 \frac{\Omega_{iv}^2 - 2(1 + \delta/3)\bar{\omega}^2}{\Omega_{iv}(\Omega_{iv}^2 - \Omega_{sc}^2)} \mu_N^2. \quad (31)$$

Using relations (27) in formulae (30) and (31), we obtain the expression for the $B(M1)$ values valid for both excitations

$$B(M1)_\nu = 2 | \langle \nu | \hat{F}_{11}^p | 0 \rangle |^2 = \frac{1 - \alpha}{8\pi} \frac{m\bar{\omega}^2}{\hbar} Q_{00} \delta^2 \frac{\Omega_\nu^2 - 2(1 + \delta/3)\bar{\omega}^2}{\Omega_\nu[\Omega_\nu^2 - \bar{\omega}^2(2 - \alpha)(1 + \delta/3)]} \mu_N^2. \quad (32)$$

Taking into account the relation $Q_{00}^0 \frac{m\omega_0}{\hbar} \simeq \frac{1}{2} \left(\frac{3}{2} A \right)^{4/3}$, which is usually [16] used to fix the value of the harmonic oscillator frequency ω_0 , we obtain the following estimate for the transition probability of the scissors mode:

$$B(M1) \uparrow = 2 | \langle sc | \hat{F}_{11}^p | 0 \rangle |^2 = \frac{(3/2)^{11/6}}{16\pi} A^{4/3} \delta \mu_N^2 = 0.042 A^{4/3} \delta \mu_N^2,$$

which practically coincides with the result of [11]: $B(M1) \uparrow = 0.043 A^{4/3} \delta \mu_N^2$, obtained with the help of the microscopic approach based on the evaluation of the sum rules.

Concluding this section one should mention, that all magnetic and electric modes of the considered model satisfy the energy weighted sum rule [1], the contribution of the spurious (or Goldstone) mode being nonzero. It is interesting to compare the contributions of the scissors mode and the spurious mode. The scissors mode (for small δ) yields:

$$2E_{sc} | \langle sc | \hat{F}_{21}^p | 0 \rangle |^2 \simeq \frac{5}{128\pi} e^2 \frac{\hbar^2}{m} Q_{00} \delta^2. \quad (33)$$

The spurious mode yields:

$$2\hbar\Omega_0 | < \Omega_0 | \hat{F}_{21}^p | 0 > |^2 = \frac{5}{8\pi} e^2 \frac{\hbar^2}{m} Q_{00} \frac{\delta^2}{1 + \delta/3}. \quad (34)$$

It is seen that the contribution of the spurious mode is approximately 16 times larger than the one of the scissors mode. This is a very significant number demonstrating the importance of excluding the spurious state from the theoretical results. Indeed, to describe correctly such a subtle phenomenon as the scissors mode, it is compulsory to eliminate the errors from spurious motion whose value can be an order of magnitude larger than the phenomenon under consideration.

3 Random Phase Approximation (RPA)

In this section we now want to derive the analogous equations for energies and transition probabilities from standard RPA theory. RPA equations in the notation of [4] are

$$\begin{aligned} \sum_{n,j} \{ [\delta_{ij}\delta_{mn}(\epsilon_m - \epsilon_i) + \bar{v}_{mjn}] X_{nj} + \bar{v}_{mni} Y_{nj} \} &= \hbar\Omega X_{mi}, \\ \sum_{n,j} \{ \bar{v}_{ijmn} X_{nj} + [\delta_{ij}\delta_{mn}(\epsilon_m - \epsilon_i) + \bar{v}_{inmj}] Y_{nj} \} &= -\hbar\Omega Y_{mi}. \end{aligned} \quad (35)$$

According to the definition of the schematic model in [4], the matrix elements of the residual interaction corresponding to the Hamiltonian (6) are written as

$$\bar{v}_{mjn} = \kappa_{\tau\tau'} \mathcal{Q}_{im}^{\tau*} \mathcal{Q}_{jn}^{\tau'}$$

with $\mathcal{Q}_{im} \equiv \langle i | q_{21} | m \rangle$ and $\kappa_{nn} = \kappa_{pp} = \kappa$, $\kappa_{np} = \bar{\kappa}$. This interaction distinguishes between protons and neutrons, so we have to introduce the isospin indices τ, τ' into the set of RPA equations (35):

$$\begin{aligned} (\epsilon_m^\tau - \epsilon_i^\tau) X_{mi}^\tau + \sum_{n,j,\tau'} \kappa_{\tau\tau'} \mathcal{Q}_{im}^{\tau*} \mathcal{Q}_{jn}^{\tau'} X_{nj}^{\tau'} + \sum_{n,j,\tau'} \kappa_{\tau\tau'} \mathcal{Q}_{im}^{\tau*} \mathcal{Q}_{nj}^{\tau'} Y_{nj}^{\tau'} &= \hbar\Omega X_{mi}^\tau, \\ \sum_{n,j,\tau'} \kappa_{\tau\tau'} \mathcal{Q}_{mi}^{\tau*} \mathcal{Q}_{jn}^{\tau'} X_{nj}^{\tau'} + (\epsilon_m^\tau - \epsilon_i^\tau) Y_{mi}^\tau + \sum_{n,j,\tau'} \kappa_{\tau\tau'} \mathcal{Q}_{mi}^{\tau*} \mathcal{Q}_{nj}^{\tau'} Y_{nj}^{\tau'} &= -\hbar\Omega Y_{mi}^\tau. \end{aligned} \quad (36)$$

The solution of these equations is

$$X_{mi}^\tau = \frac{\mathcal{Q}_{im}^{\tau*}}{\hbar\Omega - \epsilon_{mi}^\tau} K^\tau, \quad Y_{mi}^\tau = -\frac{\mathcal{Q}_{mi}^{\tau*}}{\hbar\Omega + \epsilon_{mi}^\tau} K^\tau \quad (37)$$

with $\epsilon_{mi}^\tau = \epsilon_m^\tau - \epsilon_i^\tau$ and $K^\tau = \sum_{\tau'} \kappa_{\tau\tau'} C^{\tau'}$.

The constant C^τ is defined as $C^\tau = \sum_{n,j} (\mathcal{Q}_{jn}^\tau X_{nj}^\tau + \mathcal{Q}_{nj}^\tau Y_{nj}^\tau)$. Using here the expressions for X_{nj}^τ and Y_{nj}^τ given above, one derives the useful relation

$$C^\tau = 2S^\tau K^\tau = 2S^\tau \sum_{\tau'} \kappa_{\tau\tau'} C^{\tau'}, \quad (38)$$

where the following notation is introduced:

$$S^\tau = \sum_{mi} |\mathcal{Q}_{mi}^\tau|^2 \frac{\epsilon_{mi}^\tau}{E^2 - (\epsilon_{mi}^\tau)^2} \quad (39)$$

with $E = \hbar\Omega$. Let us write out the relation (38) in detail

$$\begin{aligned} C^n - 2S^n(\kappa C^n + \bar{\kappa} C^p) &= 0, \\ C^p - 2S^p(\bar{\kappa} C^n + \kappa C^p) &= 0. \end{aligned} \quad (40)$$

The condition for existence of a nontrivial solution of this set of equations gives the secular equation

$$(1 - 2S^n \kappa)(1 - 2S^p \bar{\kappa}) - 4S^n S^p \bar{\kappa}^2 = 0. \quad (41)$$

Making obvious linear combinations of the two equations in (40), we write them in terms of isoscalar and isovector constants $C = C^n + C^p$, $\bar{C} = C^n - C^p$

$$\begin{aligned} C - 2(S^n + S^p)\kappa_0 C - 2(S^n - S^p)\kappa_1 \bar{C} &= 0, \\ \bar{C} - 2(S^n - S^p)\kappa_0 C - 2(S^n + S^p)\kappa_1 \bar{C} &= 0. \end{aligned} \quad (42)$$

Approximation (11) allows us to decouple the equations for isoscalar and isovector constants. Really, in this case $S^n = S^p \equiv S/2$; hence, we obtain two secular equations

$$1 - 2S\kappa_0 = 0, \quad \text{or} \quad 1 - S\kappa = S\bar{\kappa} \quad (43)$$

in the isoscalar case and

$$1 - 2S\kappa_1 = 0, \quad \text{or} \quad 1 - S\kappa = -S\bar{\kappa} \quad (44)$$

in the isovector one, the difference of both lies in the strength constants only. Having in mind the relation $\kappa_1 = \alpha\kappa_0$, we come to the conclusion that it is sufficient to analyze the isovector case only – the results for isoscalar one are obtained by assuming $\alpha = 1$.

3.1 Eigenfrequencies

The detailed expression for the isovector secular equation is

$$\frac{1}{2\kappa_1} = \sum_{mi} |\mathcal{Q}_{mi}|^2 \frac{\epsilon_{mi}}{E^2 - \epsilon_{mi}^2}. \quad (45)$$

The operator \mathcal{Q} has only two types of nonzero matrix elements \mathcal{Q}_{mi} in the deformed oscillator basis. Matrix elements of the first type couple states of the same major shell. All corresponding transition energies are degenerate: $\epsilon_m - \epsilon_i = \hbar(\omega_x - \omega_z) \equiv \epsilon_0$. Matrix elements of the second type couple states of the different major shells with $\Delta N = 2$. All corresponding transition energies are degenerate too: $\epsilon_m - \epsilon_i = \hbar(\omega_x + \omega_z) \equiv \epsilon_2$. Therefore, the secular equation can be rewritten as

$$\frac{1}{2\kappa_1} = \frac{\epsilon_0 \mathcal{Q}_0}{E^2 - \epsilon_0^2} + \frac{\epsilon_2 \mathcal{Q}_2}{E^2 - \epsilon_2^2}. \quad (46)$$

The sums $\mathcal{Q}_0 = \sum_{mi(\Delta N=0)} |\mathcal{Q}_{mi}|^2$ and $\mathcal{Q}_2 = \sum_{mi(\Delta N=2)} |\mathcal{Q}_{mi}|^2$ can be calculated analytically (see Appendix B):

$$\mathcal{Q}_0 = \frac{Q_{00}}{m\bar{\omega}^2} \epsilon_0, \quad \mathcal{Q}_2 = \frac{Q_{00}}{m\bar{\omega}^2} \epsilon_2. \quad (47)$$

Let us transform the secular equation (46) in polynomial form

$$E^4 - E^2[(\epsilon_0^2 + \epsilon_2^2) + 2\kappa_1(\epsilon_0 \mathcal{Q}_0 + \epsilon_2 \mathcal{Q}_2)] + [\epsilon_0^2 \epsilon_2^2 + 2\kappa_1 \epsilon_0 \epsilon_2 (\epsilon_0 \mathcal{Q}_2 + \epsilon_2 \mathcal{Q}_0)] = 0.$$

Using here the expressions (47) for \mathcal{Q}_0 , \mathcal{Q}_2 and the self-consistent value of the strength constant (A.3), we find

$$E^4 - E^2(1 - \alpha/2)(\epsilon_0^2 + \epsilon_2^2) + (1 - \alpha)\epsilon_0^2 \epsilon_2^2 = 0,$$

or

$$\Omega^4 - \Omega^2(2 - \alpha)\omega_+^2 + (1 - \alpha)\omega_-^4 = 0, \quad (48)$$

with the notation $\omega_+^2 = \omega_x^2 + \omega_z^2$ and $\omega_-^4 = (\omega_x^2 - \omega_z^2)^2$. This result coincides with that of [2].

By a trivial rearrangement of the terms in (48) one obtains the useful relation

$$\Omega^2(\Omega^2 - \omega_+^2) = (1 - \alpha)(\Omega^2 \omega_+^2 - \omega_-^4). \quad (49)$$

Inserting expressions (A.3) for ω_x^2 , ω_z^2 into (48), we find $\omega_+^2 = 2\bar{\omega}^2(1 + \delta/3)$, $\omega_-^4 = 4\delta^2\bar{\omega}^4$ and reproduce formula (18) for the isovector case

$$\Omega^4 - 2\Omega^2\bar{\omega}^2(2 - \alpha)(1 + \delta/3) + 4\bar{\omega}^4(1 - \alpha)\delta^2 = 0.$$

Taking here $\alpha = 1$ we reproduce formula (14) for the isoscalar case

$$\Omega^4 - 2\Omega^2\bar{\omega}^2(1 + \delta/3) = 0.$$

3.2 B(E2)-factors

According to [4], the transition probability for the one-body operator $\hat{F} = \sum_{s=1}^A \hat{f}_s$ is calculated by means of the formulae

$$\langle 0|\hat{F}^\tau|\nu\rangle = \sum_{mi}(f_{im}^\tau X_{mi}^{\tau,\nu} + f_{mi}^\tau Y_{mi}^{\tau,\nu}), \quad \langle \nu|\hat{F}^\tau|0\rangle = \sum_{mi}(f_{mi}^\tau X_{mi}^{\tau,\nu} + f_{im}^\tau Y_{mi}^{\tau,\nu}). \quad (50)$$

Quadrupole excitations are described by the operator (23) with $\hat{f}_{2\mu} = er^2 Y_{2\mu} = \tilde{e}Q$, where $\tilde{e} = e\sqrt{\frac{5}{16\pi}}$. The expressions for X_{mi}^τ, Y_{mi}^τ are given by formulae (37). Combining these results we get

$$\langle 0|\hat{F}_{21}^p|\nu\rangle = 2\tilde{e}K_\nu^p \sum_{mi} |\mathcal{Q}_{mi}^p|^2 \frac{\epsilon_{mi}^p}{E_\nu^2 - (\epsilon_{mi}^p)^2} = 2\tilde{e}K_\nu^p S_\nu^p = \tilde{e}C_\nu^p. \quad (51)$$

The constant C_ν^p is determined by the normalization condition

$$\delta_{\nu,\nu'} = \sum_{mi,\tau} (X_{mi}^{\tau,\nu*} X_{mi}^{\tau,\nu'} - Y_{mi}^{\tau,\nu*} Y_{mi}^{\tau,\nu'}),$$

that gives

$$\frac{1}{(C_\nu^p)^2} = E_\nu \sum_{mi} \left[\frac{|\mathcal{Q}_{mi}^p|^2}{(S_\nu^p)^2} \frac{\epsilon_{mi}^p}{[E_\nu^2 - (\epsilon_{mi}^p)^2]^2} + \frac{(C_\nu^n)^2}{(C_\nu^p)^2} \frac{|\mathcal{Q}_{mi}^n|^2}{(S_\nu^n)^2} \frac{\epsilon_{mi}^n}{[E_\nu^2 - (\epsilon_{mi}^n)^2]^2} \right]. \quad (52)$$

The ratio C^n/C^p is determined by any of the equations (40):

$$\frac{C^n}{C^p} = \frac{1 - 2S^p\kappa}{2S^p\bar{\kappa}} = \frac{2S^n\bar{\kappa}}{1 - 2S^n\kappa}. \quad (53)$$

Formula (52) is considerably simplified by the approximation (11), when $S^p = S^n \equiv S/2$, $\epsilon_{mi}^p = \epsilon_{mi}^n$, $\mathcal{Q}_{mi}^p = \mathcal{Q}_{mi}^n$. Applying the second forms of formulae (43,44) it is easy to find that in this case $C^n/C^p = \pm 1$. As a result, the final expression for $B(E2)$ value is

$$B(E2)_\nu = 2|\langle 0|\hat{F}_{21}^p|\nu\rangle|^2 = 2\tilde{e}^2 \left(16E_\nu\kappa_1^2 \sum_{mi} |\mathcal{Q}_{mi}|^2 \frac{\epsilon_{mi}}{(E_\nu^2 - \epsilon_{mi}^2)^2} \right)^{-1}. \quad (54)$$

With the help of formulae (47) this expression can be transformed into

$$\begin{aligned} B(E2)_\nu &= \frac{5}{8\pi} \frac{e^2 Q_{00}}{m\bar{\omega}^2 \alpha^2 E_\nu} \left[\frac{\epsilon_0^2}{(E_\nu^2 - \epsilon_0^2)^2} + \frac{\epsilon_2^2}{(E_\nu^2 - \epsilon_2^2)^2} \right]^{-1} \\ &= \frac{5}{16\pi} \frac{e^2 \hbar Q_{00}}{m\bar{\omega}^2 \Omega_\nu} \frac{(\Omega_\nu^2 \omega_+^2 - \omega_-^4)^2}{\Omega_\nu^4 \omega_+^2 - 2\Omega_\nu^2 \omega_-^4 + \omega_+^2 \omega_-^4}. \end{aligned} \quad (55)$$

At first sight, this expression has nothing in common with (28). Nevertheless, it can be shown that they are identical. To this end, we analyze carefully the denominator of the last expression in (55). Summing it with the secular equation (48) (multiplied by ω_+^2), which obviously does not change its value, we find after elementary combinations

$$\begin{aligned}
\text{Denom} &= \Omega_\nu^4 \omega_+^2 - 2\Omega_\nu^2 \omega_-^4 + \omega_+^2 \omega_-^4 + \omega_+^2 [\Omega_\nu^4 - \Omega_\nu^2 (2 - \alpha) \omega_+^2 + (1 - \alpha) \omega_-^4] \\
&= \omega_+^2 \Omega_\nu^2 [2\Omega_\nu^2 - (2 - \alpha) \omega_+^2] - \omega_-^4 [2\Omega_\nu^2 - (2 - \alpha) \omega_+^2] \\
&= (\Omega_\nu^2 \omega_+^2 - \omega_-^4) [2\Omega_\nu^2 - (2 - \alpha) \omega_+^2].
\end{aligned} \tag{56}$$

This result allows us to write the final expression as

$$B(E2)_\nu = \frac{5}{16\pi} \frac{e^2 \hbar}{m \bar{\omega}^2} Q_{00} \frac{\Omega_\nu^2 \omega_+^2 - \omega_-^4}{\Omega_\nu [2\Omega_\nu^2 - (2 - \alpha) \omega_+^2]}, \tag{57}$$

which coincides with (28) (we recall that $\omega_+^2 = 2\bar{\omega}^2(1 + \delta/3)$, $\omega_-^4 = 4\delta^2 \bar{\omega}^4$). By simple transformations this formula is reduced to the result of Hamamoto and Nazarewicz [2] (taking into account, that they published it without the constant factor $\frac{5}{32\pi} \frac{e^2 \hbar}{m \omega_0} Q_{00}^0$).

3.3 B(M1)-factors

In accordance with formulae (29), (50), (37) the magnetic transition matrix element is given by

$$\langle 0 | \hat{F}_{11}^p | \nu \rangle = K_\nu^p \sum_{mi} \left[\frac{(\hat{f}_{11}^p)_{im} \mathcal{Q}_{im}^{p*}}{E_\nu - \epsilon_{mi}^p} - \frac{(\hat{f}_{11}^p)_{mi} \mathcal{Q}_{mi}^{p*}}{E_\nu + \epsilon_{mi}^p} \right]. \tag{58}$$

As it is shown in Appendix B, the matrix element $(\hat{f}_{11}^p)_{im}$ is proportional to \mathcal{Q}_{im}^p (formula (B.16)). So, expression (58) is reduced to

$$\begin{aligned}
\langle 0 | \hat{F}_{11}^p | \nu \rangle &= -K_\nu^p \frac{\tilde{e} \hbar}{2c\sqrt{5}} (\omega_x^2 - \omega_z^2)^p \sum_{mi} \left[\frac{\mathcal{Q}_{im}^p \mathcal{Q}_{im}^{p*}}{\epsilon_{im}^p (E_\nu - \epsilon_{mi}^p)} - \frac{\mathcal{Q}_{mi}^p \mathcal{Q}_{mi}^{p*}}{\epsilon_{mi}^p (E_\nu + \epsilon_{mi}^p)} \right] \\
&= K_\nu^p \frac{\tilde{e} \hbar}{c\sqrt{5}} (\omega_x^2 - \omega_z^2)^p E_\nu \sum_{mi} \frac{|\mathcal{Q}_{mi}^p|^2}{\epsilon_{mi}^p [E_\nu^2 - (\epsilon_{mi}^p)^2]}.
\end{aligned} \tag{59}$$

With the help of approximation (11) and the expressions (47) for \mathcal{Q}_0 , \mathcal{Q}_2 we find

$$\begin{aligned}
\langle 0 | \hat{F}_{11}^p | \nu \rangle &= \frac{C_\nu^p}{2S_\nu^p} \frac{\tilde{e} \hbar}{c\sqrt{5}} (\omega_x^2 - \omega_z^2) \frac{Q_{00}}{2m\bar{\omega}^2} \left(\frac{E_\nu}{E_\nu^2 - \epsilon_0^2} + \frac{E_\nu}{E_\nu^2 - \epsilon_2^2} \right) \\
&= -2\kappa_1 C_\nu^p \frac{\tilde{e}}{c\sqrt{5}} (\omega_x^2 - \omega_z^2) \frac{Q_{00}}{m\bar{\omega}^2} \frac{\Omega_\nu (\Omega_\nu^2 - \omega_+^2)}{\alpha (\Omega_\nu^2 \omega_+^2 - \omega_-^4)} \\
&= \frac{C_\nu^p}{2} \frac{\tilde{e}}{c\sqrt{5}} (\omega_x^2 - \omega_z^2) \frac{1 - \alpha}{\Omega_\nu}.
\end{aligned} \tag{60}$$

Relation (49) and the self-consistent value of the strength constant $\kappa_1 = \alpha\kappa_0$ were used in the last step. For the magnetic transition probability we have

$$B(M1)_\nu = 2| \langle 0 | \hat{F}_{11}^p | \nu \rangle |^2 = 2 \frac{(C_\nu^p)^2}{4} \frac{\tilde{e}^2}{5c^2} \omega_-^4 \frac{(1-\alpha)^2}{\Omega_\nu^2} = \frac{\omega_-^4}{20c^2} \frac{(1-\alpha)^2}{\Omega_\nu^2} B(E2)_\nu. \quad (61)$$

This relation between $B(M1)$ and $B(E2)$ was also found (up to the factor $1/(20c^2)$) by Hamamoto and Nazarewicz [2]. Substituting expression (57) for $B(E2)$ into (61) we reproduce (with the help of relation (49)) formula (32).

3.4 “Synthetic” scissors and spurious state

The nature of collective excitations calculated with the method of Wigner function moments is quite easily revealed analyzing the roles of collective variables describing the phenomenon. The solution of this problem in the RPA approach is not so obvious. That is why the nature of the low-lying states has often been established by considering overlaps of these states with the “pure scissors state” [18, 19] or “synthetic state” [2] produced by the action of the scissors operator which antirotates proton versus neutron distributions

$$\hat{S}_x = \mathcal{N}^{-1}(\langle I_x^{n2} \rangle \hat{I}_x^p - \langle I_x^{p2} \rangle \hat{I}_x^n)$$

on the ground state

$$|Syn\rangle = \hat{S}_x |0\rangle.$$

In the considered model the overlap of the “synthetic” state with the real scissors mode (and with IVGQR) can be calculated analytically. Let us at first modify the definition of the “synthetic” state. Due to axial symmetry one can use the \hat{I}_y^τ component instead of \hat{I}_x^τ , or any of their linear combinations, for example, the $\mu = 1$ component of the magnetic operator $\hat{F}_{1\mu}^\tau$, which is much more convenient for us. The terms $\langle I_x^{\tau 2} \rangle$ are introduced to ensure the orthogonality of the synthetic scissors to the spurious state $|Sp\rangle = (\hat{I}^n + \hat{I}^p)|0\rangle$. However, we do not need these terms because the collective states $|\nu\rangle$ of our model are already orthogonal to $|Sp\rangle$ (see below); hence, the overlaps $\langle Syn|\nu\rangle$ will be free from any admixtures of $|Sp\rangle$. So, we use the following definitions of the synthetic and spurious states:

$$|Syn\rangle = \mathcal{N}^{-1}(\hat{F}_{11}^p - \hat{F}_{11}^n)|0\rangle, \quad |Sp\rangle = (\hat{F}_{11}^p + \hat{F}_{11}^n)|0\rangle.$$

Let us demonstrate the orthogonality of the spurious state to all the rest of the states $|\nu\rangle$. As the first step it is necessary to show that the secular equation (41) has the solution $E = 0$. We need the expression for $S^\tau(E = 0) \equiv S^\tau(0)$. In accordance with (39), we have

$$S^\tau(E) = \left[\frac{\epsilon_0 \mathcal{Q}_0}{E^2 - \epsilon_0^2} + \frac{\epsilon_2 \mathcal{Q}_2}{E^2 - \epsilon_2^2} \right]^\tau, \quad S^\tau(0) = - \left[\frac{\mathcal{Q}_0}{\epsilon_0} + \frac{\mathcal{Q}_2}{\epsilon_2} \right]^\tau.$$

The expressions for \mathcal{Q}_0^τ , \mathcal{Q}_2^τ are easily extracted from formulae (B.10), (B.11):

$$\mathcal{Q}_0^\tau = \frac{\hbar}{m} Q_{00}^\tau \left[\frac{1 + \frac{4}{3}\delta}{\omega_x} - \frac{1 - \frac{2}{3}\delta}{\omega_z} \right]^\tau, \quad \mathcal{Q}_2^\tau = \frac{\hbar}{m} Q_{00}^\tau \left[\frac{1 + \frac{4}{3}\delta}{\omega_x} + \frac{1 - \frac{2}{3}\delta}{\omega_z} \right]^\tau. \quad (62)$$

So we find

$$\begin{aligned} S^\tau(0) &= -\frac{\hbar}{m} Q_{00}^\tau \left[\frac{1 + \frac{4}{3}\delta}{\omega_x} \left(\frac{1}{\epsilon_2} + \frac{1}{\epsilon_0} \right) + \frac{1 - \frac{2}{3}\delta}{\omega_z} \left(\frac{1}{\epsilon_2} - \frac{1}{\epsilon_0} \right) \right]^\tau \\ &= -\frac{\hbar^2}{m} \frac{4\delta^\tau Q_{00}^\tau}{\epsilon_2^\tau \epsilon_0^\tau} = -\frac{1}{m} \frac{3Q_{20}^\tau}{(\omega_x^2 - \omega_z^2)^\tau}, \end{aligned} \quad (63)$$

where, in accordance with (B.12),

$$(\omega_x^2 - \omega_z^2)^p = -\frac{6}{m} (\kappa Q_{20}^p + \bar{\kappa} Q_{20}^n), \quad (\omega_x^2 - \omega_z^2)^n = -\frac{6}{m} (\kappa Q_{20}^n + \bar{\kappa} Q_{20}^p). \quad (64)$$

Finally, we get

$$\begin{aligned} 2S^p(0) &= \frac{Q_{20}^p}{\kappa Q_{20}^p + \bar{\kappa} Q_{20}^n}, \quad 1 - 2S^p(0)\kappa = \frac{\bar{\kappa} Q_{20}^n}{\kappa Q_{20}^p + \bar{\kappa} Q_{20}^n}, \\ 2S^n(0) &= \frac{Q_{20}^n}{\kappa Q_{20}^n + \bar{\kappa} Q_{20}^p}, \quad 1 - 2S^n(0)\kappa = \frac{\bar{\kappa} Q_{20}^p}{\kappa Q_{20}^n + \bar{\kappa} Q_{20}^p}. \end{aligned}$$

It is easy to see that substituting these expressions into (41) we obtain an identity; therefore, the secular equation has a zero energy solution.

For the second step it is necessary to calculate the overlap $\langle Sp | \nu \rangle$. Summing (59) with an analogous expression for neutrons, we get

$$\begin{aligned} \langle Sp | \nu \rangle &= \frac{\tilde{e}\hbar}{c\sqrt{5}} E_\nu \sum_\tau K_\nu^\tau (\omega_x^2 - \omega_z^2)^\tau \sum_{mi} \frac{|\mathcal{Q}_{mi}^\tau|^2}{\epsilon_{mi}^\tau (E_\nu^2 - \epsilon_{mi}^2)^\tau} \\ &= \frac{\tilde{e}\hbar}{c\sqrt{5}} E_\nu \sum_\tau K_\nu^\tau (\omega_x^2 - \omega_z^2)^\tau \sum_{mi} \frac{|\mathcal{Q}_{mi}^\tau|^2 \epsilon_{mi}^\tau}{(\epsilon_{mi}^2)^\tau (E_\nu^2 - \epsilon_{mi}^2)^\tau}. \end{aligned} \quad (65)$$

Applying the algebraical identity

$$\frac{1}{\epsilon^2(E^2 - \epsilon^2)} = \frac{1}{E^2} \left(\frac{1}{\epsilon^2} + \frac{1}{E^2 - \epsilon^2} \right)$$

and remembering the definition (39) of S^τ we rewrite (65) as

$$\begin{aligned} \langle Sp|\nu \rangle &= \frac{\tilde{\hbar}}{c\sqrt{5}E_\nu} \sum_\tau K_\nu^\tau (\omega_x^2 - \omega_z^2)^\tau (S^\tau - S^\tau(0)) \\ &= \frac{\tilde{\hbar}}{c\sqrt{5}} \frac{K_\nu^p}{E_\nu} \left[(\omega_x^2 - \omega_z^2)^p (S^p - S^p(0)) + (\omega_x^2 - \omega_z^2)^n (S^n - S^n(0)) \frac{K_\nu^n}{K_\nu^p} \right]. \end{aligned} \quad (66)$$

In accordance with (38) and (53),

$$\frac{K_\nu^n}{K_\nu^p} = \frac{1 - 2S^p\kappa}{2S^n\bar{\kappa}}. \quad (67)$$

Noting now (see formula (63)) that $(\omega_x^2 - \omega_z^2)^\tau S^\tau(0) = -\frac{3}{m}Q_{20}^\tau$ and taking into account relations (64), we find

$$\begin{aligned} \langle Sp|\nu \rangle &= \beta \left\{ [(\kappa Q_2^p + \bar{\kappa} Q_2^n) 2S^p - Q_2^p] + [(\kappa Q_2^n + \bar{\kappa} Q_2^p) 2S^n - Q_2^n] \frac{1 - 2S^p\kappa}{2S^n\bar{\kappa}} \right\} \\ &= \beta \left\{ [(2S^p\kappa - 1)Q_2^p + 2S^p\bar{\kappa}Q_2^n] + [(2S^n\kappa - 1)Q_2^n + 2S^n\bar{\kappa}Q_2^p] \frac{1 - 2S^p\kappa}{2S^n\bar{\kappa}} \right\} \\ &= \beta \left\{ 2S^p\bar{\kappa}Q_2^n + (2S^n\kappa - 1)Q_2^n \frac{1 - 2S^p\kappa}{2S^n\bar{\kappa}} \right\} \\ &= \beta \frac{Q_2^n}{2S^n\bar{\kappa}} \{ 2S^n\bar{\kappa}2S^p\bar{\kappa} - (1 - 2S^n\kappa)(1 - 2S^p\kappa) \} = 0, \end{aligned} \quad (68)$$

where $\beta = -\frac{3}{m} \frac{\tilde{\hbar}}{c\sqrt{5}} \frac{K_\nu^p}{E_\nu}$ and $Q_2 \equiv Q_{20}$. The expression in the last curly brackets coincides obviously with the secular equation (41) that proves the orthogonality of the spurious state to all physical states of the considered model. So we can conclude that strictly speaking this is not a spurious state, but one of the exact eigenstates of the model corresponding to the integral of motion $I^n + I^p$. The same conclusion was made by N. Lo Iudice [20] who solved this problem approximately with the help of several assumptions (the small deformation limit, for example).

The problem of the "spurious" state being solved, the calculation of overlaps $\langle Syn|\nu \rangle$ becomes trivial. Really, we have shown that $\langle 0|\hat{F}_{11}^n + \hat{F}_{11}^p|\nu \rangle = 0$, hence $\langle 0|\hat{F}_{11}^n|\nu \rangle = -\langle 0|\hat{F}_{11}^p|\nu \rangle$. Then $\langle Syn|\nu \rangle = \mathcal{N}^{-1} \langle 0|\hat{F}_{11}^p - \hat{F}_{11}^n|\nu \rangle = 2\mathcal{N}^{-1} \langle 0|\hat{F}_{11}^p|\nu \rangle$ and

$$U^2 \equiv |\langle Syn|\nu \rangle|^2 = 2\mathcal{N}^{-2} B(M1)_\nu. \quad (69)$$

The nontrivial part of the problem is the calculation of the normalization factor \mathcal{N} . It is important not to forget about the time dependence of the synthetic state which should be determined by the external field:

$$|Syn(t)\rangle = \mathcal{N}^{-1} [(\hat{F}_{11}^p - \hat{F}_{11}^n)e^{-i\Omega t} + (\hat{F}_{11}^p - \hat{F}_{11}^n)^\dagger e^{i\Omega t}] |0\rangle.$$

Then we have

$$\begin{aligned}
\mathcal{N}^2 &= 2 < 0 | (\hat{F}_{11}^p - \hat{F}_{11}^n)^\dagger (\hat{F}_{11}^p - \hat{F}_{11}^n) | 0 > \\
&= 2 \sum_{ph} < 0 | (\hat{F}_{11}^p - \hat{F}_{11}^n)^\dagger | ph > < ph | (\hat{F}_{11}^p - \hat{F}_{11}^n) | 0 > = 2 \sum_{ph} | < ph | (\hat{F}_{11}^p - \hat{F}_{11}^n) | 0 > |^2 \\
&= 2 \sum_{\tau, ph} | < ph | \hat{F}_{11}^\tau | 0 > |^2 = 2 \sum_{\tau, ph} |(f_{11}^\tau)_{ph}|^2.
\end{aligned} \tag{70}$$

With the help of relation (B.16) we find

$$\begin{aligned}
\mathcal{N}^2 &= \frac{2}{5} \left(\frac{e\hbar}{2c} \right)^2 \sum_{\tau, ph} \left(\omega_-^4 \frac{| < ph | r^2 Y_{21} | 0 > |^2}{\epsilon_{ph}^2} \right)^\tau \\
&= \frac{1}{8\pi} \left(\frac{e\hbar}{2c} \right)^2 \sum_{\tau} (\omega_-^4)^\tau \left(\frac{\mathcal{Q}_0}{\epsilon_0^2} + \frac{\mathcal{Q}_2}{\epsilon_2^2} \right)^\tau.
\end{aligned} \tag{71}$$

Expressions for \mathcal{Q}_0^τ , \mathcal{Q}_2^τ , ω_x^τ , ω_z^τ are given by formulae (62), (B.12). To get a definite number, it is necessary to make some assumption concerning the relation between neutron and proton equilibrium characteristics. As usual, we apply the approximation (11), i.e., suppose $Q_{00}^n = Q_{00}^p$, $Q_{20}^n = Q_{20}^p$. It is easy to check that in this case formulae for $\omega_{x,z}^\tau$ are reduced to the ones for the isoscalar case, namely (A.3), and $\mathcal{Q}_0^\tau = \mathcal{Q}_0/2$, $\mathcal{Q}_2^\tau = \mathcal{Q}_2/2$, where \mathcal{Q}_0 and \mathcal{Q}_2 are given by (47). So we get

$$\mathcal{N}^2 = \frac{\omega_-^4}{8\pi} \left(\frac{e\hbar}{2c} \right)^2 \frac{Q_{00}}{m\bar{\omega}^2} \left(\frac{1}{\epsilon_0} + \frac{1}{\epsilon_2} \right) = \frac{\delta}{2\pi} \frac{m\omega_x}{\hbar} Q_{00} \mu_N^2. \tag{72}$$

The estimation of the overlap for ^{156}Gd with $\delta = 0.27$ gives $\mathcal{N}^2 = 34.72\mu_N^2$ and $U^2 = 0.53$ (see eq. (69)), that is two times larger than the result of [18] obtained in QRPA calculations with the Skyrme forces. The disagreement can naturally be attributed to the difference in forces and especially to the lack of pair correlations in our approach.

In the small deformation limit $U^2 = \frac{1}{2}\sqrt{\frac{3}{2}} \approx 0.6$. This is the maximum possible overlap of the "pure" (or "synthetic") scissors with the real scissors. Increasing δ and /or taking into account pairing correlations decreases its value, that is confirmed by numerous microscopic calculations with various forces [21]. Such small overlap leads inevitably to the conclusion, that the original model of counter rotating rigid rotors [22] has not very much in common with the real scissors mode, the correct description of which requires the proper treatment of the Fermi surface deformation and the coupling with IVGQR.

3.5 Superdeformation

A certain drawback of our approach is that, so far, we have not included the superfluidity into our description. On the other hand, our formulae (20, 32) can be successfully used for the description of superdeformed nuclei where the pairing is very weak [2, 22]. For example, applying them to the superdeformed nucleus ^{152}Dy ($\delta \simeq 0.6$, $\hbar\omega_0 = 41/A^{1/3}\text{MeV}$), we get

$$E_{iv} = 20.8 \text{ MeV}, \quad B(M1)_{iv} = 15.9 \mu_N^2$$

for the isovector GQR and

$$E_{sc} = 4.7 \text{ MeV}, \quad B(M1)_{sc} = 20.0 \mu_N^2$$

for the scissors mode. There are not many results of other calculations to compare with. As a matter of fact, there are only two papers considering this problem.

The phenomenological TRM model [22] predicts

$$E_{iv} \simeq 26 \text{ MeV}, \quad B(M1)_{iv} \simeq 26 \mu_N^2, \quad E_{sc} \simeq 6.1 \text{ MeV}, \quad B(M1)_{sc} \simeq 22 \mu_N^2.$$

The only existing microscopic calculation [2] in the framework of QRPA with separable forces gives

$$E_{iv} \simeq 28 \text{ MeV}, \quad B(M1)_{iv} \simeq 37 \mu_N^2, \quad E_{sc} \simeq 5 - 6 \text{ MeV}, \quad B(M1)_{1+} \simeq 23 \mu_N^2.$$

Here $B(M1)_{1+}$ denotes the total $M1$ orbital strength carried by the calculated $K^\pi = 1^+$ QRPA excitations modes in the energy region below 20 MeV.

It is easy to see that in the case of IVGQR one can speak, at least, about qualitative agreement. Our results for E_{sc} and $B(M1)_{sc}$ are in good agreement with that of phenomenological model and with E_{sc} and $B(M1)_{1+}$ of Hamamoto and Nazarewicz.

It is possible to extract from the histogram of [2] the value of the overlap of calculated low-lying 1^+ excitations with the synthetic scissors state: $|\langle \text{Syn} | 1^+ \rangle|^2 \approx 0.4$. The result of our calculation $U^2 = 0.43$ agrees with it very well. So, the comparison of our calculations with that of QRPA shows, that we have excellent agreement in superdeformed nuclei and rather large disagreement in moderately deformed nuclei. On the other hand it is known [2], that

pairing is very weak at the superdeformation and becomes important at moderate deformations. Therefore, as a consequence the correct treatment of pair correlations is important for an accurate description of the scissors mode. This shall be the subject of future work.

4 WFM versus RPA

In this section we want to establish the precise relation between the WFM and RPA methods. Though it can be guessed that working in both cases with the full configuration space, i.e. with a complete set of particle and hole states in the case of RPA and with all moments of all ranks in the case of WFM, identical results shall be obtained, it is still important to study the details of this relation because the behaviour of WFM and RPA under truncation of the space turns out to be radically different. The exact relations between the RPA amplitudes X_{ph} and Y_{ph} and the respective WFM variables can be established with the help of the linear response theory.

Let us consider the response of the system to a weak external time-dependent field

$$\hat{W}(t) = \hat{W} \exp(-i\Omega t) + \hat{W}^\dagger \exp(i\Omega t) \quad (73)$$

with $\hat{W} = \sum_{kq} w_{kq} a_k^\dagger a_q$. The linear response is then given by [4]:

$$\rho_{kq}^{(1)}(t) = \sum_{k'q'} \left[R_{kq,k'q'}(\Omega) e^{-i\Omega t} + R_{qk,k'q'}^*(\Omega) e^{i\Omega t} \right] w_{k'q'},$$

where

$$R_{kq,k'q'}(\Omega) = \sum_{\nu} \left(\frac{\langle 0 | a_q^\dagger a_k | \nu \rangle \langle \nu | a_{k'}^\dagger a_{q'} | 0 \rangle}{\hbar(\Omega - \Omega_\nu)} - \frac{\langle 0 | a_{k'}^\dagger a_{q'} | \nu \rangle \langle \nu | a_q^\dagger a_k | 0 \rangle}{\hbar(\Omega + \Omega_\nu)} \right)$$

is the RPA response function [4] and the index pairs kq and $k'q'$ are restricted to particle hole and hole particle pairs. Indices p, q include spin and isospin quantum numbers σ and τ . For the change of the average value of an arbitrary operator we have:

$$\delta \langle \Psi | \hat{F} | \Psi \rangle = \sum_{kq} f_{qk} \rho_{kq}^{(1)}. \quad (74)$$

We now are ready to analyze the WFM variables. The first one is

$$R_{\lambda\mu}^\tau(t) = 2(2\pi\hbar)^{-3} \int d^3p \int d^3r r r_{\lambda\mu}^2 f^\tau(\mathbf{r}, \mathbf{p}, t).$$

Using here definitions of the Wigner function, the δ -function and the density matrix [4]

$$\rho(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma'\tau') = \sum_{kq} \phi_k^*(\mathbf{r}'\sigma'\tau') \phi_q(\mathbf{r}\sigma\tau) < \Psi | a_k^\dagger a_q | \Psi > \quad (75)$$

we find

$$\begin{aligned} R_{\lambda\mu}^\tau(t) &= \frac{2}{(2\pi\hbar)^3} \int d^3r r_{\lambda\mu}^2 \int d^3s \int d^3p \exp(-i\mathbf{p} \cdot \mathbf{s}/\hbar) \rho^\tau(\mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{r} - \frac{\mathbf{s}}{2}, t) \\ &= 2 \int d^3r r_{\lambda\mu}^2 \rho^\tau(\mathbf{r}, \mathbf{r}, t) = \sum_\sigma \int d^3r r_{\lambda\mu}^2 \rho(\mathbf{r}\sigma\tau, \mathbf{r}\sigma\tau, t) \\ &= \sum_{kq} \sum_\sigma \int d^3r r_{\lambda\mu}^2 \phi_k^*(\mathbf{r}\sigma\tau) \phi_q(\mathbf{r}\sigma\tau) < \Psi | a_k^\dagger a_q | \Psi > \\ &= \sum_{kq} (r_{\lambda\mu}^2)_{kq}^\tau < \Psi | a_k^\dagger a_q | \Psi > = < \Psi | \sum_{kq} (r_{\lambda\mu}^2)_{kq}^\tau a_k^\dagger a_q | \Psi > \\ &= < \Psi | \sum_{s=1}^{N_\tau} r_{\lambda\mu}^2(s) | \Psi > = < \Psi | \hat{R}_{\lambda\mu}^\tau | \Psi >, \end{aligned} \quad (76)$$

i.e. this is just the ground state expectation value of the operator $\hat{R}_{\lambda\mu} = \sum_{s=1}^A (r_s^2)_{\lambda\mu}$. In accordance with (74) the variation of this variable is

$$\begin{aligned} \delta R_{\lambda\mu}^\tau(t) &= \mathcal{R}_{\lambda\mu}^\tau(t) = \sum_{kq} (r_{\lambda\mu}^2)_{kq}^\tau \rho_{qk}^{(1)}(t) \\ &= \sum_\nu (< 0 | \hat{R}_{\lambda\mu}^\tau | \nu > c_\nu - < \nu | \hat{R}_{\lambda\mu}^\tau | 0 > \bar{c}_\nu) e^{-i\Omega t} \\ &\quad + \sum_\nu (< \nu | \hat{R}_{\lambda\mu}^\tau | 0 > c_\nu^* - < 0 | \hat{R}_{\lambda\mu}^\tau | \nu > \bar{c}_\nu^*) e^{i\Omega t}, \end{aligned} \quad (77)$$

where

$$c_\nu = \frac{< \nu | \hat{W} | 0 >}{\hbar(\Omega - \Omega_\nu)} = \sum_{kq} \frac{< \nu | a_k^\dagger a_q | 0 >}{\hbar(\Omega - \Omega_\nu)} w_{kq}, \quad \bar{c}_\nu = \frac{< 0 | \hat{W} | \nu >}{\hbar(\Omega + \Omega_\nu)} = \sum_{kq} \frac{< 0 | a_k^\dagger a_q | \nu >}{\hbar(\Omega + \Omega_\nu)} w_{kq}. \quad (78)$$

Equation (77) demonstrates in an obvious way the structure of the variables $\delta R_{\lambda\mu}$. They are linear combinations of the transition matrix elements $< 0 | \hat{R}_{\lambda\mu} | \nu >$ which are, in turn, linear combinations of the RPA amplitudes X_{kq}, Y_{kq} . Introducing the notation $\hat{L}_{\lambda\mu} = \sum_{s=1}^A (r_s \hat{p}_s)_{\lambda\mu}$ and

$\hat{P}_{\lambda\mu} = \sum_{s=1}^A (\hat{p}_s^2)_{\lambda\mu}$ we can proceed in a similar way with

$$L_{\lambda\mu}^\tau(t) = 2(2\pi\hbar)^{-3} \int d^3p \int d^3r (rp)_{\lambda\mu} f^\tau(\mathbf{r}, \mathbf{p}, t)$$

and

$$P_{\lambda\mu}^\tau(t) = 2(2\pi\hbar)^{-3} \int d^3p \int d^3r p_{\lambda\mu}^2 f^\tau(\mathbf{r}, \mathbf{p}, t).$$

Inserting the expressions for $\delta R_{\lambda\mu}, \delta L_{\lambda\mu}$ into the first equation of the set (10) we find

$$-i\Omega \sum_\nu (< 0|\hat{R}_{\lambda\mu}|\nu > c_\nu - < \nu|\hat{R}_{\lambda\mu}|0 > \bar{c}_\nu) = \frac{2}{m} \sum_\nu (< 0|\hat{L}_{\lambda\mu}|\nu > c_\nu - < \nu|\hat{L}_{\lambda\mu}|0 > \bar{c}_\nu).$$

It is sufficient to consider only the part with the $e^{-i\Omega t}$ time dependence. Multiplying this equation by $(\Omega - \Omega_\nu)$ and taking the limit $\Omega \rightarrow \Omega_\nu$ we find the equation

$$-i\Omega_\nu < 0|\hat{R}_{\lambda\mu}|\nu > = \frac{2}{m} < 0|\hat{L}_{\lambda\mu}|\nu >, \quad (79)$$

which can be called as the dynamical equation for the transition matrix element $< 0|\hat{R}_{\lambda\mu}|\nu >$. Naturally, in the same way the dynamical equations for transition matrix elements $< 0|\hat{L}_{\lambda\mu}|\nu >$ and $< 0|\hat{P}_{\lambda\mu}|\nu >$ can be extracted from the second and third (linearized) equations of (10).

Now we can show, that exactly the same dynamical equations for Transition Matrix Elements (TME) can be derived from RPA equations. To this end we combine the RPA equations (36) with the definition (50) of matrix elements:

$$\hbar\Omega_\nu \sum_{mi} (f_{im}^\tau X_{mi}^{\tau,\nu} + f_{mi}^\tau Y_{mi}^{\tau,\nu}) = \sum_{mi} \epsilon_{mi} (f_{im}^\tau X_{mi}^{\tau,\nu} - f_{mi}^\tau Y_{mi}^{\tau,\nu}) + K_\nu^\tau \sum_{mi} (f_{im}^\tau \mathcal{Q}_{im}^{\tau*} - f_{mi}^\tau \mathcal{Q}_{mi}^{\tau*}). \quad (80)$$

Taking into account the relations

$$\epsilon_{mi} f_{im} = [\hat{f}, H_0]_{im}, \quad \epsilon_{mi} f_{mi} = -[\hat{f}, H_0]_{mi},$$

one rewrites this equation as

$$\hbar\Omega_\nu < 0|\hat{F}^\tau|\nu > = \sum_{mi} \{[\hat{f}^\tau, H_0^\tau]_{im} X_{mi}^{\tau,\nu} + [\hat{f}^\tau, H_0^\tau]_{mi} Y_{mi}^{\tau,\nu} + K_\nu^\tau (f_{im}^\tau \mathcal{Q}_{im}^{\tau*} - f_{mi}^\tau \mathcal{Q}_{mi}^{\tau*})\}. \quad (81)$$

The Hamiltonian of the axially deformed harmonic oscillator corresponding to the mean field (8) is

$$H_0^\tau(\mathbf{r}) = \sum_{s=1}^{N_\tau} \left\{ \frac{\hat{\mathbf{p}}_s^2}{2m} + \frac{1}{2} m \omega^2 \mathbf{r}_s^2 + Z_{20}^\tau(eq) r_{20}^2(s) \right\}. \quad (82)$$

Let us consider the operator $\hat{f} = \sqrt{6} r_{21}^2 = q_{21} = \mathcal{Q}$. Calculating the commutator

$$[r_{21}^2, H_0] = i\hbar \frac{2}{m} (r\hat{p})_{21}$$

we find from (81) the following equation

$$\begin{aligned} \hbar\Omega_\nu < 0 | \sum_{s=1}^{N_\tau} (r_{21}^2)_s^\tau | \nu > &= i\hbar \frac{2}{m} \sum_{mi} \{ ((r\hat{p})_{21})_{im}^\tau X_{mi}^{\tau,\nu} + ((r\hat{p})_{21})_{mi}^\tau Y_{mi}^{\tau,\nu} \} \\ &+ K_\nu^\tau \sum_{mi} (\mathcal{Q}_{im}^\tau \mathcal{Q}_{im}^{\tau*} - \mathcal{Q}_{mi}^\tau \mathcal{Q}_{mi}^{\tau*}). \end{aligned} \quad (83)$$

Taking into account relations $(\mathcal{Q}^*)_{im} = (\mathcal{Q})_{mi}^*$ and $|\mathcal{Q}_{mi}|^2 = |\mathcal{Q}_{im}|^2$ we find, that the last sum in (83) is equal to zero. Applying again formula (50) we write (83) as

$$-i\Omega_\nu < 0 | \hat{R}_{21}^\tau | \nu > = \frac{2}{m} < 0 | \hat{L}_{21}^\tau | \nu > \quad (84)$$

reproducing equation (79). The dynamical equations for $< 0 | \hat{L}_{21}^\tau | \nu >$ and $< 0 | \hat{P}_{21}^\tau | \nu >$ are obtained in a similar way by considering the operators $\hat{f} = (r\hat{p})_{21}^2$ and $\hat{f} = \hat{p}_{21}^2$ respectively. As it could be expected, the resulting equations reproduce the corresponding TME equations derived from (10) with the help of the limit procedure.

So, there exists one-to-one correspondence between the set of dynamical equations for WFM variables and the set of dynamical equations for Transition Matrix Elements (TME). This correspondence makes obvious the fact that both sets have the same eigenvalues. On the other hand the TME equations are just linear combinations of the RPA equations. Therefore we can conclude that RPA and WFM approaches generate identical eigenvalues. In this sense both approaches are equivalent in all aspects. This concerns for instance also the transition probabilities. However, for this equivalence to be exact, one needs to work in the full space in both approaches, that is in the complete particle hole space in RPA and taking all phase space moments of all powers in WFM, a task which can hardly be tackled in general.

The difference of the two approaches then shows up if truncations of the dimension of the equations have to be operated. In RPA one usually solves the equations with a restricted number of discrete particle hole pairs, i.e. the dimension of the RPA matrix is finite (in some works the RPA equations for finite nuclei are, however, solved in full space, including continuum states [23, 24]). The result of such a diagonalisation usually yields a huge number of discrete eigenvalues approximating more or less the spectrum one would obtain from a solution in the full space. For instance resonances in the continuum (e.g. giant resonances) will be mocked up by a bunch of discrete states whose envelope may simulate the full solution. Reducing the

dimension of the particle hole space too much may lead to a situation where the full solution is only approximated rather badly and in an uncontrolled manner.

In the WFM method the dynamical equations for Cartesian tensors of the rank $n = 2$ are coupled (by the interaction terms in (5)) with dynamical equations for tensors of the rank $n = 3$, these equations being coupled with the ones for tensors of the rank $n = 4$ and so on up to $n = \infty$. Here one hopes that the essential part of physics is described by a small number of the lowest ranks tensors. The hope is based on the assumption that the higher rank tensors (moments) are responsible for the more refined details and that neglecting them does not appreciably influence the description of the more global physics which is described with the lower ranks tensors. This assumption is substantiated in past applications of the WFM method to realistic situations with Skyrme forces for the description of collective nuclear modes [6, 25, 26]. In those works it has indeed been demonstrated that even with a very limited number of low rank phase space moments one can faithfully reproduce the centroid position of the collective states. In this sense the WFM method is rather similar to the sum rule approach which works, however, only in the cases when practically all strength is exhausted by one state, whereas WFM method works also in situations when the strength is distributed among a few excitations. From these studies it is then permitted to assume that the inclusion of higher and higher rank moments will just give raise to a refinement of the gross structure obtained with the low rank tensors. A formal convergence study of this type has been performed in the infinite matter case [27] where it was indeed shown that the moment method allows to approach the full solution in an optimized way.

The net result is, that WFM and RPA approximate the exact infinite spectrum into a finite number of eigenfrequencies with, however, different convergence.

An analogous situation occurs with transition probabilities. Let us analyze, for example, the expression (77) for the WFM variable $\delta R_{\lambda\mu}^\tau(t) = \mathcal{R}_{\lambda\mu}^\tau(t)$. Using the definition (78) of c_ν with the external field operator $\hat{W} = \hat{R}_{\lambda\mu}^{\tau\dagger}$, we find

$$\begin{aligned} \delta R_{\lambda\mu}^\tau(t) &= \sum_{\nu=1}^{N_c} \left(\frac{\langle 0 | \hat{R}_{\lambda\mu}^\tau | \nu \rangle \langle \nu | \hat{R}_{\lambda\mu}^{\tau\dagger} | 0 \rangle}{\hbar(\Omega - \Omega_\nu)} - \frac{\langle \nu | \hat{R}_{\lambda\mu}^\tau | 0 \rangle \langle 0 | \hat{R}_{\lambda\mu}^{\tau\dagger} | \nu \rangle}{\hbar(\Omega + \Omega_\nu)} \right) e^{-i\Omega t} \\ &= \sum_{\nu=1}^{N_c} \left(\frac{|\langle 0 | \hat{R}_{\lambda\mu}^\tau | \nu \rangle|^2}{\hbar(\Omega - \Omega_\nu)} - \frac{|\langle 0 | \hat{R}_{\lambda\mu}^{\tau\dagger} | \nu \rangle|^2}{\hbar(\Omega + \Omega_\nu)} \right) e^{-i\Omega t}. \end{aligned} \quad (85)$$

The summation limit N_c depends on the method of calculation. In the case of the exact solution $N_c = \infty$, for RPA N_c is usually of the order of several *hundreds or thousands*, for WFM N_c usually is not more than around *a dozen*. Naturally, the eigenvalues Ω_ν and eigenstates $|\nu\rangle$ are different in each case. So, the strength, that in RPA was distributed over hundreds or thousands levels, in WFM is concentrated only on several levels, i.e. averaging of levels is accompanied by the redistribution of the strength. The variable $\delta R_{\lambda\mu}^\tau(t)$ is the quantum mechanical observable, so its value should not depend on the basis $|\nu\rangle$. Hence, the right hand sides of (85), calculated by two methods, should coincide if both methods are mutually consistent. This statement can e.g. be checked with the help of sum rules. Generally in RPA sum rules are well fulfilled for a sufficiently large particle hole space which in realistic cases can become quite significant, whereas in WFM sum rules are generally already well fulfilled even with a small number of low rank moments (see e.g.[6, 25, 26, 28]).

The essential difference between WFM and RPA methods lies in their practical use. The RPA equations (35) are constructed in such a way that the increase in dimension does not cause any formal problems and finally it is only a question of computer power what dimension can be handled. Quite on the contrary the increase of dimension in WFM is a nontrivial task. Beyond a certain order of the moments even the reduction of Cartesian tensors to the irreducible ones becomes a very difficult task. However, the spirit of WFM is rather to reproduce the gross structure of a couple of prominent collective states, a situation which it can handle very efficiently. However, in cases where there is strong fragmentation, direct diagonalisation of the standard RPA equations is more efficient.

In conclusion WFM and RPA are equivalent when the full particle hole configuration space in RPA and the infinite number of moments in WFM are considered. However, under truncation of the spaces both methods have different convergence properties. In the general case for WFM only a few moments are sufficient to get the correct gross structure of the collective part of the spectrum, whereas in RPA one in general must take into account a quite large configuration space to produce reasonable results.

5 Green's function method

One of the important subjects of comparing RPA and WFM methods are the current distributions. The WFM method, a priori, can not give the exact results, because it deals only with integrals over the whole phase space. It would therefore be very interesting to evaluate the accuracy of this approximation by comparing it with the exact result. Unfortunately, even for the simple model HO+QQ it is impossible to derive in RPA closed analytical expressions for currents of the scissors mode and IVGQR. That is why we consider in this section the Green's Function (GF) method, which allows one to find explicit expressions for the currents directly.

Following the paper of H. Kohl, P. Schuck and S. Stringari [29] we will consider at first the isoscalar case. Conserving on the right hand side of equation (4) only the first term of the sin-function expansion leads to the Vlasov equation

$$\frac{\partial f}{\partial t} = \nabla H_W \cdot \nabla^p f - \nabla^p H_W \cdot \nabla f. \quad (86)$$

In our case the Wigner transform H_W coincides with the classical Hamiltonian H_c . Having in mind small amplitude vibrations we have to linearize (86): $f = f_0 + f_1$, $H_c = H_0 + H_1$, with f_0 being the solution of the time independent equation. The linearized version of (86) is

$$\frac{\partial f_1}{\partial t} + \nabla^p H_0 \cdot \nabla f_1 - \nabla H_0 \cdot \nabla^p f_1 = S(\mathbf{r}, \mathbf{p}, t), \quad (87)$$

where $S(\mathbf{r}, \mathbf{p}, t) = \nabla H_1 \cdot \nabla^p f_0$. This equation will be solved with the Green's function method.

We have

$$\left(\frac{\partial}{\partial t} + \nabla^p H_0 \cdot \nabla - \nabla H_0 \cdot \nabla^p\right) G^{(t-t')}(\mathbf{r}\mathbf{p}, \mathbf{r}'\mathbf{p}') = \delta(\mathbf{r} - \mathbf{r}')\delta(\mathbf{p} - \mathbf{p}')\delta(t - t') \quad (88)$$

with [29]

$$G^{(t-t')}(\mathbf{r}\mathbf{p}, \mathbf{r}'\mathbf{p}') = \delta[\mathbf{r}_c(\mathbf{r}, \mathbf{p}, t' - t) - \mathbf{r}']\delta[\mathbf{p}_c(\mathbf{r}, \mathbf{p}, t' - t) - \mathbf{p}']\theta(t - t'),$$

where $\mathbf{r}_c(\mathbf{r}, \mathbf{p}, t' - t)$, $\mathbf{p}_c(\mathbf{r}, \mathbf{p}, t' - t)$ are solutions of classical equations of motion with initial conditions \mathbf{r}, \mathbf{p} . The solution of (87) can be written as

$$\begin{aligned} f_1(\mathbf{r}, \mathbf{p}, t) &= f_1^h + \int_{-\infty}^{\infty} dt' \int d^3r' d^3p' G^{(t-t')}(\mathbf{r}\mathbf{p}, \mathbf{r}'\mathbf{p}') S(\mathbf{r}', \mathbf{p}', t') \\ &= f_1^h + \int_{-\infty}^t dt' S(\mathbf{r}_c, \mathbf{p}_c, t'), \end{aligned} \quad (89)$$

where f_1^h is the solution of the homogeneous equation. It is obvious that any function of variables \mathbf{r}_c and \mathbf{p}_c satisfies the homogeneous equation, however it does not play any role at resonance and we therefore will omit it in the forthcoming.

We consider the axially deformed harmonic oscillator with the quadrupole–quadrupole residual interaction V_{res} . Therefore the single-particle Hamiltonian is

$$H_0 = \frac{p^2}{2m} + \frac{m}{2}[\omega_x^2(x^2 + y^2) + \omega_z^2 z^2]$$

and $H_1 = V_{res}$.

We are interested in the part of the residual interaction with $|\mu| = 1$. In accordance with formula (A.1) it can be written as

$$V_{res} = -\kappa_0[Q_{21}(t)q_{2-1}(\mathbf{r}) + Q_{2-1}(t)q_{21}(\mathbf{r})] = 12\kappa_0 Q_1(t)[xz + yz]$$

with

$$\begin{aligned} Q_1(t) &= 2 \int d\{\mathbf{p}, \mathbf{r}\} f(\mathbf{r}, \mathbf{p}, t) xz = 2 \int d\{\mathbf{p}, \mathbf{r}\} f(\mathbf{r}, \mathbf{p}, t) yz \\ &= 2 \int d\{\mathbf{p}, \mathbf{r}\} [f_0(\mathbf{r}, \mathbf{p}) + f_1(\mathbf{r}, \mathbf{p}, t)] xz = 2 \int d\{\mathbf{p}, \mathbf{r}\} f_1(\mathbf{r}, \mathbf{p}, t) xz. \end{aligned}$$

With the help of the Thomas-Fermi approximation for the static distribution function

$$f_0 = \theta(\epsilon_F - H_0)$$

the right hand side of (87) is found to be

$$S(\mathbf{r}, \mathbf{p}, t) = -12 \frac{\kappa_0}{m} Q_1(t) \delta(\epsilon_F - H_0) [p_x z + p_z x + p_y z + p_z y].$$

The classical trajectories are determined by the solution of Hamilton equations

$\dot{r}_{c,i} = \frac{\partial H_0}{\partial p_i}$, $\dot{p}_{c,i} = -\frac{\partial H_0}{\partial r_i}$ with $i = x, y, z$. In our case they are

$$r_{c,i}(t) = r_i \cos \omega_i t + \frac{p_i}{m\omega_i} \sin \omega_i t, \quad p_{c,i}(t) = p_i \cos \omega_i t - m\omega_i r_i \sin \omega_i t.$$

Formula (89) then gives

$$\begin{aligned} f_1(\mathbf{r}, \mathbf{p}, t) &= -6 \frac{\kappa_0}{m} \delta(\epsilon_F - H_0) \int_{-\infty}^t dt' Q_1(t') \times \\ &\quad \times \left\{ \frac{1}{\omega_x} (p_x + p_y) z [\omega_+ \cos \omega_+(t' - t) + \omega_- \cos \omega_-(t' - t)] \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\omega_z} p_z (x + y) [\omega_+ \cos \omega_+ (t' - t) - \omega_- \cos \omega_- (t' - t)] \\
& + \frac{1}{m \omega_x \omega_z} (p_x + p_y) p_z [\omega_+ \sin \omega_+ (t' - t) - \omega_- \sin \omega_- (t' - t)] \\
& - m(x + y) z [\omega_+ \sin \omega_+ (t' - t) + \omega_- \sin \omega_- (t' - t)],
\end{aligned} \tag{90}$$

where $\omega_{\pm} = \omega_x \pm \omega_z$.

So, we have derived a complicated integral equation for the perturbed distribution function which may not easily be solved in general. As a matter of fact the analytic possibilities of the Green's function method are, without further consideration, exhausted at this point. However, expressions (88) and (89) point to the possibility to use the so-called pseudo particle method [30], in case the classical trajectories are not known analytically.

In order to proceed to the evaluation of the eigenfrequencies and transition probabilities we again apply the method of moments. Integrating (90) over the whole phase space with the weights xz , $p_x p_z$, $z p_x + x p_z$ and $z p_x - x p_z$ we obtain the following set of coupled integral equations

$$\begin{aligned}
Q_1(t) &= \beta \int_{-\infty}^t dt' Q_1(t') [\omega_+ \sin \omega_+ (t' - t) + \omega_- \sin \omega_- (t' - t)], \\
P_1(t) &= -\beta m^2 \omega_x \omega_z \int_{-\infty}^t dt' Q_1(t') [\omega_+ \sin \omega_+ (t' - t) - \omega_- \sin \omega_- (t' - t)], \\
L_1(t) &= -\beta m \int_{-\infty}^t dt' Q_1(t') [\omega_+^2 \cos \omega_+ (t' - t) + \omega_-^2 \cos \omega_- (t' - t)], \\
I_y(t) &= -\beta m \omega_+ \omega_- \int_{-\infty}^t dt' Q_1(t') [\cos \omega_+ (t' - t) + \cos \omega_- (t' - t)],
\end{aligned} \tag{91}$$

where

$$\begin{aligned}
\beta &= \frac{2\kappa_0 \pi^3 \epsilon_F^4}{m^2 \omega_x^4 \omega_z^3} \frac{4}{(2\pi\hbar)^3} \\
&= 12\kappa_0 \int d\{\mathbf{p}, \mathbf{r}\} x^2 z^2 \delta(\epsilon_F - H_0) = \frac{12\kappa_0}{m^4 \omega_x^2 \omega_z^2} \int d\{\mathbf{p}, \mathbf{r}\} p_x^2 p_z^2 \delta(\epsilon_F - H_0) \\
&= \frac{12\kappa_0}{m^2 \omega_z^2} \int d\{\mathbf{p}, \mathbf{r}\} x^2 p_z^2 \delta(\epsilon_F - H_0) = \frac{12\kappa_0}{m^2 \omega_x^2} \int d\{\mathbf{p}, \mathbf{r}\} z^2 p_x^2 \delta(\epsilon_F - H_0)
\end{aligned}$$

and the following notation is introduced

$$P_1(t) = 2 \int d\{\mathbf{p}, \mathbf{r}\} f_1(\mathbf{r}, \mathbf{p}, t) p_x p_z, \quad L_1(t) = 2 \int d\{\mathbf{p}, \mathbf{r}\} f_1(\mathbf{r}, \mathbf{p}, t) (z p_x + x p_z),$$

$$I_y(t) = 2 \int d\{\mathbf{p}, \mathbf{r}\} f_1(\mathbf{r}, \mathbf{p}, t) (zp_x - xp_z).$$

By simple means these equations are reduced to a set of differential equations. At first we perform time derivatives of all equations in (91):

$$\begin{aligned} \dot{Q}_1(t) &= -\beta \int_{-\infty}^t dt' Q_1(t') [\omega_+^2 \cos \omega_+(t' - t) + \omega_-^2 \cos \omega_-(t' - t)], \\ \dot{P}_1(t) &= \beta m^2 \omega_x \omega_z \int_{-\infty}^t dt' Q_1(t') [\omega_+^2 \cos \omega_+(t' - t) - \omega_-^2 \cos \omega_-(t' - t)], \\ \dot{L}_1(t) &= -\beta m \left\{ (\omega_+^2 + \omega_-^2) Q_1(t) + \int_{-\infty}^t dt' Q_1(t') [\omega_+^3 \sin \omega_+(t' - t) + \omega_-^3 \sin \omega_-(t' - t)] \right\}, \\ \dot{I}_y(t) &= -\beta m \omega_+ \omega_- \left\{ 2Q_1(t) + \int_{-\infty}^t dt' Q_1(t') [\omega_+ \sin \omega_+(t' - t) + \omega_- \sin \omega_-(t' - t)] \right\}. \end{aligned} \quad (92)$$

Solving (91) with respect of four time integrals (containing $\sin \omega_{\pm}$ and $\cos \omega_{\pm}$) we can substitute found expressions into (92). We obtain

$$\begin{aligned} \dot{Q}_1(t) &= \frac{1}{m} L_1(t), \\ \dot{L}_1(t) &= -m(2\beta + 1)(\omega_x^2 + \omega_z^2) Q_1(t) + \frac{2}{m} P_1(t), \\ \dot{P}_1(t) &= -\frac{m}{2} [(\omega_x^2 + \omega_z^2) L_1(t) - (\omega_x^2 - \omega_z^2) I_y(t)], \\ \dot{I}_y(t) &= -m(2\beta + 1) \omega_+ \omega_- Q_1(t). \end{aligned} \quad (93)$$

Due to conservation of the angular momentum the right hand side of the last equation must be equal to zero. So we have the requirement

$$2\beta + 1 = 0, \quad \text{or} \quad \kappa_0 = -\frac{m^2 \omega_x^4 \omega_z^3 (2\pi\hbar)^3}{4\pi^3 \epsilon_F^4} \frac{4}{4}. \quad (94)$$

With the help of the relation

$$A\langle r^2 \rangle = 2 \int d\{\mathbf{p}, \mathbf{r}\} r^2 f_0 = 2 \int d\{\mathbf{p}, \mathbf{r}\} r^2 \theta(\epsilon_F - H_0) = \frac{\pi^3 \epsilon_F^4 (\omega_x^2 + 2\omega_z^2)}{3m\omega_x^4 \omega_z^3} \frac{4}{(2\pi\hbar)^3} \quad (95)$$

and formulae (A.3) for ω_x, ω_z the expression for κ_0 is reduced to

$$\kappa_0 = -\frac{m(\omega_x^2 + 2\omega_z^2)}{12A\langle r^2 \rangle} = -\frac{m\bar{\omega}^2}{4A\langle r^2 \rangle}, \quad (96)$$

which is just the familiar expression for the self-consistent value of the strength constant (see Appendix A). This is a rather interesting result, because the well known formula is obtained without the usual self consistency requirement [31]. As it is known, in the absence of external fields the angular momentum of any system is conserved. The short range interparticle interactions depending on the module of the interparticle distance $|\mathbf{r}_i - \mathbf{r}_j|$ create the scalar, i.e. rotational invariant, mean field, which exactly repeats the shape of the nucleus. When we imitate the mean field by a rotational invariant function, the angular momentum will be conserved independently of the shape of this function due to a pure mathematical reason: angular momentum operator commutes with a scalar field. If we use the non rotational invariant function (as in our case), mathematics does not help and the shape of the function becomes important. If the function does not follow exactly the shape of the system, the latter will react on this inconsistency as on the external field, that leads to the nonconservation of an angular momentum. Therefore the requirement of the angular momentum conservation in this case becomes equivalent to the requirement of the self consistency. This is seen very well in the method of moments. Integrating equation (87) over the phase space with the weight $zp_x - xp_z$ we obtain the dynamical equation for I_y

$$\frac{d}{dt}I_y = m(\omega_z^2 - \omega_x^2)Q_1 + \alpha(\langle x^2 \rangle - \langle z^2 \rangle)Q_1. \quad (97)$$

The requirement of the angular momentum conservation gives the following relation

$$m(\omega_z^2 - \omega_x^2) = \alpha(\langle z^2 \rangle - \langle x^2 \rangle). \quad (98)$$

Obviously it is the requirement of the consistency between the shapes of the potential and the nucleus. In principle this relation is less restrictive than the standard self consistency requirement [31]. However, the latter satisfies equation (98) what can be easily checked with the help of Appendix A.

So, finally the set of equations (93) is reduced to

$$\begin{aligned} \dot{Q}_1(t) &= \frac{1}{m}L_1(t), \\ \dot{L}_1(t) &= \frac{2}{m}P_1(t), \\ \dot{P}_1(t) &= -m\bar{\omega}^2[(1 + \frac{1}{3}\delta)L_1(t) - \delta I_y(t)], \\ \dot{I}_y(t) &= 0. \end{aligned} \quad (99)$$

Taking into account the relations between the definitions of variables in (13) and (99)

$$Q_1 = -Re\mathcal{R}_{21}, \quad P_1 = -Re\mathcal{P}_{21}, \quad L_1 = -2Re\mathcal{L}_{21}$$

(which follow from formulae $r_{21}^2 = -z(x + iy)$ and $(rp)_{21} = -\frac{1}{2}[zp_x + xp_z + i(zp_y + yp_z)]$) and $I_y = 2Re\mathcal{L}_{11}$, it is easy to see, that the last set of equations is identical to (13).

With the help of relations (91) the Wigner function (89) can be written in terms of the Wigner function moments. Taking into account equations of motion (99) and the time dependence of variables via $e^{-i\Omega t}$ (which leads to the equality $I_y = 0$) one finds

$$f_1(\mathbf{r}, \mathbf{p}, t) = \frac{3\kappa_0}{\beta m^2} \delta(\epsilon_F - H_0) \left\{ -i\Omega m \left[\frac{1}{\omega_x^2} (p_x + p_y) z + \frac{1}{\omega_z^2} p_z (x + y) \right] - \left(\frac{1}{\omega_x^2} + \frac{1}{\omega_z^2} \right) (p_x + p_y) p_z + 2m^2 z (x + y) \right\} Q_1(t).$$

In the case of $\delta = 0$ it reproduces the result of [29].

Having the Wigner function one can calculate transition probabilities in the same way as in WFM method.

Let us consider now the problem with two sorts of particles: neutrons and protons. All variables and parameters acquire isotopic index τ . The part of the residual interaction with $|\mu| = 1$, in accordance with formula (8) becomes $V_1^\tau = Z_1^\tau(t)[xz + yz]$ with $Z_1^n(t) = 12(\kappa Q_1^n + \bar{\kappa} Q_1^p)$, $Z_1^p(t) = 12(\kappa Q_1^p + \bar{\kappa} Q_1^n)$ and $Q_1^\tau = \int d\{\mathbf{p}, \mathbf{r}\} f_1^\tau(\mathbf{r}, \mathbf{p}, t) xz$. The expression for the Wigner function is obtained from formula (90) by changing the factor $6\kappa Q_1(t')$ by $\frac{1}{2}Z_1^\tau(t')$. The dynamical equations for isovector variables $\bar{Q}_1 = Q_1^n - Q_1^p$, $\bar{P}_1 = P_1^n - P_1^p$, $\bar{L}_1 = L_1^n - L_1^p$, and $\bar{I}_y = I_y^n - I_y^p$ can be derived (in approximation (11)) exactly in the same way as the equations for isoscalar ones. As it is expected, they coincide with (17).

As we see, in the considered simple model all results of WFM method are identical to that of Green's Function (GF) method. Having in mind also that both methods generate the same set of dynamical equations for collective variables (Wigner function moments), one could suspect their identity. In general, this is not quite true. The principal difference between the two methods is more or less obvious. In the GF method one finds first the formal solution of eq. (5) and only afterwards one takes the phase space moments of the found Wigner function to obtain the final solution of the physical problem. In the WFM method one takes from the

beginning the phase space moments of equation (5) without any attempts to find the “natural” expression for the Wigner function.

The reason of coincidence of all results is quite simple. For the harmonic oscillator with multipole–multipole residual interaction of arbitrary rank (multipolarity) the equations of both methods can be derived without any approximations – the interaction of the multipolarity n generates the set of dynamical equations for tensors (moments) of the rank n . For the GF method this is easily seen from formula (89). In the case of the WFM method it is seen very well from the structure of equation (87). When one takes the moments of rank n , neither the left hand side nor the right hand side of this equation can generate moments of rank higher than n . The coincidence of results in the case of $n = 3$ was demonstrated in [32].

The power and simplicity of the GF method are restricted by the potentials for which the analytical solutions for classical trajectories are known. In the case of realistic forces the GF method loses its simplicity and transparency, however the pseudo particle method [30] can still be applied. The WFM method does not meet any difficulties and continues to be a convenient and powerful tool for the description of the collective motion what was demonstrated by calculations with Skyrme forces [6]. For an illustration of this property of the WFM method, currents are a good example, because the procedure of their construction with WFM is general enough to be used for any type of force (see section 6.1 below and [1]).

6 Flows

We are interested in the trajectories of infinitesimal displacements of neutrons and protons during their vibrational motion, i.e. in the lines of currents. The infinitesimal displacements are determined by the magnitudes and directions of the nucleon velocities $\mathbf{u}(\mathbf{r}, t)$, given by

$$\begin{aligned}
m\rho(\mathbf{r}, t)\mathbf{u}(\mathbf{r}, t) &= \int \frac{4d^3p}{(2\pi\hbar)^3} \mathbf{p} f(\mathbf{r}, \mathbf{p}, t) \\
&= \frac{4}{(2\pi\hbar)^3} \int d^3s \int d^3p \mathbf{p} \exp(-i\mathbf{p} \cdot \mathbf{s}/\hbar) \rho(\mathbf{r} + \frac{\mathbf{s}}{2}, \mathbf{r} - \frac{\mathbf{s}}{2}, t) \\
&= -2i\hbar \{(\nabla - \nabla')\rho(\mathbf{r}, \mathbf{r}', t)\}_{r=r'} = -\frac{i\hbar}{2} \sum_{\sigma, \tau} \{(\nabla - \nabla')\rho(\mathbf{r}\sigma\tau, \mathbf{r}'\sigma\tau, t)\}_{r=r'} \\
&= -\frac{i\hbar}{2} \sum_{pq} \sum_{\sigma, \tau} \{\phi_p^*(\mathbf{r}\sigma\tau) \nabla \phi_q(\mathbf{r}\sigma\tau) - \phi_q(\mathbf{r}\sigma\tau) \nabla \phi_p^*(\mathbf{r}\sigma\tau)\} < \Psi | a_p^\dagger a_q | \Psi >
\end{aligned}$$

$$\begin{aligned}
&= m \sum_{pq} j_{pq}(\mathbf{r}) \rho_{qp}(t) = m \langle \Psi | \sum_{pq} j_{pq}(\mathbf{r}) a_p^\dagger a_q | \Psi \rangle \\
&= m \langle \Psi | \hat{J}(\mathbf{r}) | \Psi \rangle.
\end{aligned} \tag{100}$$

The current density operator $\hat{J}(\mathbf{r})$ has the standard quantum mechanical definition [4]:

$$\begin{aligned}
\hat{J}(\mathbf{r}) &= \sum_{s=1}^A \hat{j}_s(\mathbf{r}) = -\frac{i\hbar}{2m} \sum_{s=1}^A [\delta(\mathbf{r} - \hat{\mathbf{r}}_s) \nabla_s + \nabla_s \delta(\mathbf{r} - \hat{\mathbf{r}}_s)] = \sum_{pq} j_{pq}(\mathbf{r}) a_p^\dagger a_q, \\
j_{pq}(\mathbf{r}) &= -\frac{i\hbar}{2m} \langle p | [\delta(\mathbf{r} - \hat{\mathbf{r}}) \nabla + \nabla \delta(\mathbf{r} - \hat{\mathbf{r}})] | q \rangle \\
&= \frac{i\hbar}{2m} \sum_{\sigma, \tau} [\phi_q(\mathbf{r}\sigma\tau) \nabla \phi_p^*(\mathbf{r}\sigma\tau) - \phi_p^*(\mathbf{r}\sigma\tau) \nabla \phi_q(\mathbf{r}\sigma\tau)] = 4 \frac{i\hbar}{2m} [\phi_q(\mathbf{r}) \nabla \phi_p^*(\mathbf{r}) - \phi_p^*(\mathbf{r}) \nabla \phi_q(\mathbf{r})].
\end{aligned}$$

The variation of \mathbf{u} generated by the external field (73) is

$$\begin{aligned}
\rho^{eq}(\mathbf{r}) \delta \mathbf{u}(\mathbf{r}, t) &= \sum_{pq} j_{pq}(\mathbf{r}) \rho_{qp}^{(1)}(t) \\
&= \sum_{\nu} [\langle 0 | \hat{J}(\mathbf{r}) | \nu \rangle c_{\nu} - \langle \nu | \hat{J}(\mathbf{r}) | 0 \rangle \bar{c}_{\nu}] e^{-i\Omega t} \\
&\quad + \sum_{\nu} [\langle \nu | \hat{J}(\mathbf{r}) | 0 \rangle c_{\nu}^* - \langle 0 | \hat{J}(\mathbf{r}) | \nu \rangle \bar{c}_{\nu}^*] e^{i\Omega t}.
\end{aligned} \tag{101}$$

To proceed further three options are possible.

6.1 WFM method

The first way was developed within the WFM approach [6]. It allows one to derive an approximate analytical expression for $\delta \mathbf{u}(\mathbf{r}, t)$. The main idea lies in the parametrization of infinitesimal displacements $\xi(\mathbf{r}, t)$. Let us recall the main points. By definition $\delta \mathbf{u}_i(\mathbf{r}, t) = \frac{\partial \xi_i(\mathbf{r}, t)}{\partial t}$. The displacement ξ_i is parametrized [1, 6] by the expansion

$$\xi_i(\mathbf{r}, t) = G_i(t) + \sum_{j=1}^3 G_{i,j}(t) x_j + \sum_{j,k=1}^3 G_{i,jk}(t) x_j x_k + \sum_{j,k,l=1}^3 G_{i,jkl}(t) x_j x_k x_l + \dots \tag{102}$$

which, in principle, is infinite, however one makes the approximation keeping only the first terms and neglecting the remainder. For example, in [1] only the two first terms were kept. It turned out, that the G_i do not contribute to the final results due to the triplanar symmetry of considered nuclei. The coefficients $G_{i,j}$ were expressed analytically in terms of the variables

$\bar{\mathcal{R}}_{21}(t)$ and $\bar{\mathcal{L}}_{11}(t)$. Using the dynamical relations between $\bar{\mathcal{R}}_{21}(t)$ and $\bar{\mathcal{L}}_{11}(t)$ given by the last equation of the set (17), the final formulae for $\xi_i(\mathbf{r}, t)$ were found to be

$$\xi_1 = \sqrt{2}B\bar{\mathcal{J}}_{13}x_3, \quad \xi_2 = \sqrt{2}B\bar{\mathcal{J}}_{23}x_3, \quad \xi_3 = \sqrt{2}A(\bar{\mathcal{J}}_{13}x_1 + \bar{\mathcal{J}}_{23}x_2) \quad (103)$$

with

$$\begin{aligned} \bar{\mathcal{J}}_{13} &= (\bar{\mathcal{R}}_{2-1} - \bar{\mathcal{R}}_{21})/2, \quad \bar{\mathcal{J}}_{23} = i(\bar{\mathcal{R}}_{2-1} + \bar{\mathcal{R}}_{21})/2, \\ A &= \frac{3}{\sqrt{2}}[1 - 2\frac{\bar{\omega}^2}{\Omega^2}(1 - \alpha)\delta]/[Q_{00}(1 - \frac{2}{3}\delta)], \\ B &= \frac{3}{\sqrt{2}}[1 + 2\frac{\bar{\omega}^2}{\Omega^2}(1 - \alpha)\delta]/[Q_{00}(1 + \frac{4}{3}\delta)]. \end{aligned} \quad (104)$$

By definition the infinitesimal displacements ξ_i are the differentials ($\xi_1 = dx, \xi_2 = dy, \xi_3 = dz$). This fact allows one to construct the differential equations for current fields. For example, for the current field in the plane $y = 0$ we have

$$\frac{dx}{dz} = \frac{B}{A} \frac{z}{x} \quad \longrightarrow \quad xdx - \frac{B}{A}zdz = 0. \quad (105)$$

Integrating this equation we find

$$x^2 + \sigma z^2 = \text{const} \equiv c \quad \longrightarrow \quad \frac{x^2}{c} + \frac{z^2}{c/\sigma} = 1,$$

where $\sigma = -B/A$. Depending on the sign of σ this curve will be either an ellipse or a hyperbola. It was shown in [1] that the curve is an ellipse for the scissors mode and it is a hyperbola for IVGQR (see Figs 1,2). It was shown also, that the real motion of the scissors mode is a mixture of rotational and irrotational behaviour. To get a quantitative measure for the contribution of each kind of motion, it is sufficient to write the displacement $\vec{\xi}$ as the superposition of a rotational component with the coefficient a and an irrotational one with the coefficient b [21]:

$$\vec{\xi} = a\vec{e}_x \times \vec{r} + b\nabla(yz) = a(0, -z, y) + b(0, z, y).$$

Comparing the components $\xi_y = (b - a)z$, $\xi_z = (b + a)y$ with ξ_2, ξ_3 in (103) we find

$$b - a = \sqrt{2}\bar{\mathcal{J}}_{23}B, \quad b + a = \sqrt{2}\bar{\mathcal{J}}_{23}A \quad \longrightarrow \quad a = \eta(1 + \sigma), \quad b = \eta(1 - \sigma),$$

where $\eta = \bar{\mathcal{J}}_{23}A/\sqrt{2}$. So, for the scissors mode in the small δ limit we have

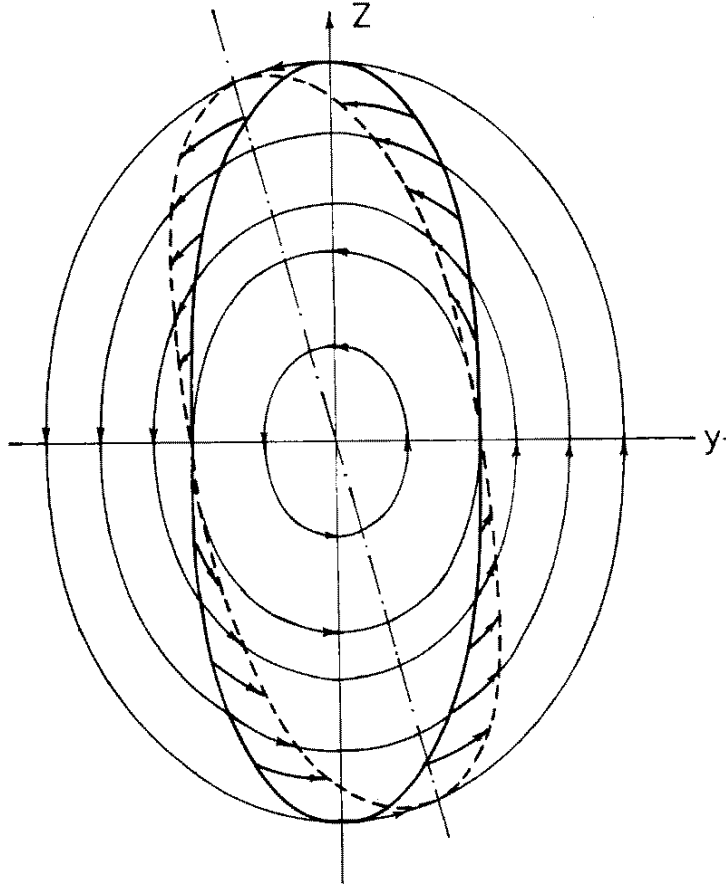


Figure 1: Schematic picture of isovector displacements for the scissors mode. Thin ellipses are the lines of currents. The thick oval is the initial position of the nucleus' surface (common for protons and neutrons). The dashed oval is the final position of the protons' (or neutrons') surface as a result of an infinitesimal displacements shown by the arrows.

$$a = 2\eta(1 - \frac{3}{4}\delta), \quad b = \frac{3}{2}\eta\delta, \quad b/a \simeq \frac{3}{4}\delta(1 + \frac{3}{4}\delta) \approx \frac{3}{4}\delta, \quad (106)$$

i.e. the current of the scissors mode is dominated by rotational motion. The contributions of the two kinds of motion to the IVGQR are

$$a = \frac{1}{2}\eta\delta, \quad b = 2\eta(1 - \frac{1}{4}\delta), \quad a/b \simeq \frac{1}{4}\delta(1 + \frac{1}{4}\delta) \approx \frac{1}{4}\delta, \quad (107)$$

i.e. the current of the IVGQR is dominated by irrotational motion.

Transitions currents are calculated in WFM analogously to transition probabilities. The pole structure of the right hand side of equation (101) tells us, that the transition current can be calculated by means of an expression similar to (22):

$$< 0 | \hat{J}_i(\mathbf{r}) | \nu > = \hbar \lim_{\Omega \rightarrow \Omega_\nu} (\Omega - \Omega_\nu) \rho^{eq}(\mathbf{r}) \overline{\dot{\xi}_i(\mathbf{r}, t) \exp(i\Omega t)} / < \nu | \hat{W} | 0 >. \quad (108)$$

For the ξ_i from above we obtain (using formulae (77) and (78))

$$\begin{aligned} < 0 | \hat{J}_3(\mathbf{r}) | \nu > &= -i\Omega_\nu \rho^{eq}(\mathbf{r}) \frac{A}{\sqrt{2}} [< 0 | \hat{R}_{2-1} - \hat{R}_{21} | \nu > x_1 + i < 0 | \hat{R}_{2-1} + \hat{R}_{21} | \nu > x_2], \\ < 0 | \hat{J}_2(\mathbf{r}) | \nu > &= \Omega_\nu \rho^{eq}(\mathbf{r}) \frac{B}{\sqrt{2}} < 0 | \hat{R}_{2-1} + \hat{R}_{21} | \nu > x_3, \\ < 0 | \hat{J}_1(\mathbf{r}) | \nu > &= -i\Omega_\nu \rho^{eq}(\mathbf{r}) \frac{B}{\sqrt{2}} < 0 | \hat{R}_{2-1} - \hat{R}_{21} | \nu > x_3. \end{aligned} \quad (109)$$

As it is seen, transition currents are proportional to transition probabilities.

If necessary, one can find the next term of the series (102). To calculate the respective coefficients $G_{i,jkl}(t)$ in the WFM method one is obliged to derive (and solve) the set of dynamical equations for higher (fourth) order moments of the Wigner function. Examples of similar calculations for third rank tensors can be found in [33].

6.2 RPA method

The procedure of constructing the flow distributions in RPA is more complicated. It is necessary at first to calculate transition currents. Having solutions (37) for X_{mi}^ν, Y_{mi}^ν one can do it with the help of formula (50):

$$\begin{aligned} < 0 | \hat{J}(\mathbf{r}) | \nu > &= \sum_{mi} (j_{im} X_{mi}^\nu + j_{mi} Y_{mi}^\nu) = K_\nu \sum_{mi} \left\{ \frac{j_{im} \mathcal{Q}_{im}^*}{E_\nu - \epsilon_{mi}} - \frac{j_{mi} \mathcal{Q}_{mi}^*}{E_\nu + \epsilon_{mi}} \right\} \\ &= K_\nu \left\{ \sum_{mi(\Delta N=0)} \left[\frac{j_{im} \mathcal{Q}_{im}^*}{E_\nu - \epsilon_0} - \frac{j_{mi} \mathcal{Q}_{mi}^*}{E_\nu + \epsilon_0} \right] + \sum_{mi(\Delta N=2)} \left[\frac{j_{im} \mathcal{Q}_{im}^*}{E_\nu - \epsilon_2} - \frac{j_{mi} \mathcal{Q}_{mi}^*}{E_\nu + \epsilon_2} \right] \right\}. \end{aligned} \quad (110)$$

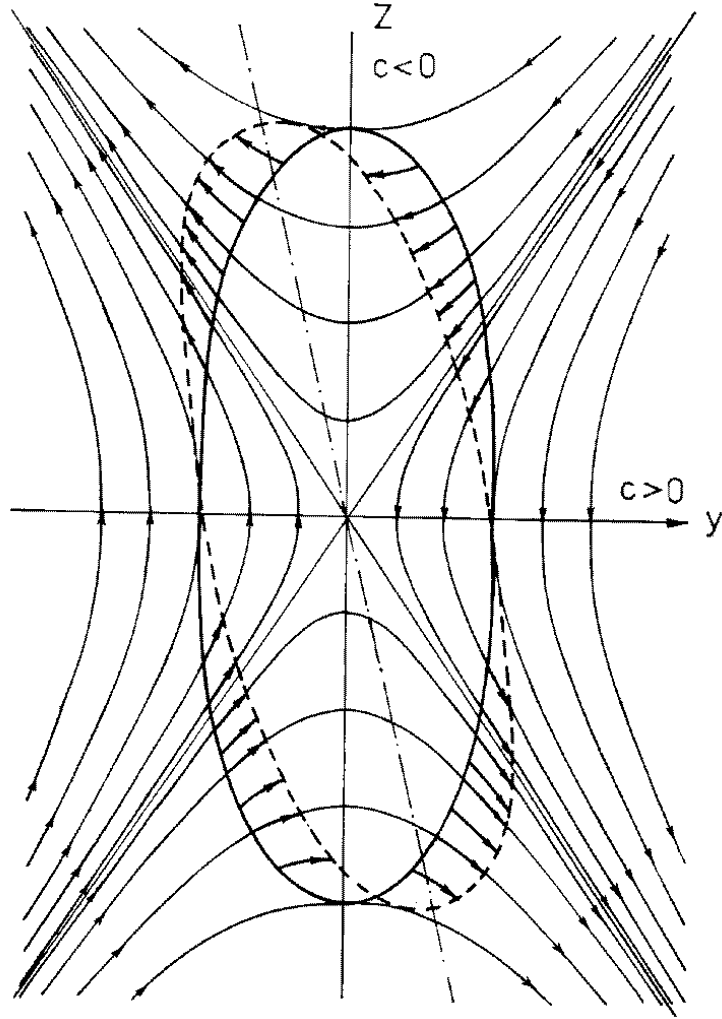


Figure 2: Schematic picture of isovector displacements for the high-lying mode (IVGQR). The lines of currents are shown by thin lines (hyperbolae). The thick oval is the initial position of the nucleus' surface (common for protons and neutrons). The dashed oval is the final position of the protons' (or neutrons') surface as a result of an infinitesimal displacements shown by the arrows.

The operator \mathcal{Q} has a finite number of particle hole matrix elements \mathcal{Q}_{mi} , so, in principle, the sums in (110) can be calculated exactly. The same is true for the coefficients c_ν (78). Therefore, in accordance with (101) one could hope to find the exact RPA result for the velocity distribution $\delta\mathbf{u}(\mathbf{r}, t)$. Unfortunately, because of the pole structure of coefficients $c_\nu(\Omega)$, it can be done for any Ω except the required frequency Ω_ν corresponding to the considered mode (resonance). Of course, it is clear that in the case of Ω close enough to Ω_ν the main contribution into $\delta\mathbf{u}$ comes from the single matrix element $\langle 0|\hat{J}(\mathbf{r})|\nu \rangle$. That is why, to get an idea about the distribution of currents in the RPA eigenstate $|\nu \rangle$ it is sufficient to know the transition matrix element $\langle 0|\hat{J}(\mathbf{r})|\nu \rangle$. However, even in this simple model one can not find a compact analytical expression for sums in (110) – the field of velocities can be constructed only numerically.

As we have already seen it is much more convenient to deal with lines of currents. The differential equation for them can be derived with the help of formula (101). With a $e^{-i\Omega t}$ time dependence we rewrite it in the more convenient form

$$-\rho^{eq}(\mathbf{r})i\Omega\xi_i(\mathbf{r}) = \sum_{\sigma}[\langle 0|\hat{J}(\mathbf{r})|\sigma \rangle c_{\sigma} - \langle \sigma|\hat{J}(\mathbf{r})|0 \rangle \bar{c}_{\sigma}].$$

and define the ratio

$$\frac{\xi_1(\mathbf{r})}{\xi_3(\mathbf{r})} = \frac{\sum_{\sigma}[\langle 0|\hat{J}_1(\mathbf{r})|\sigma \rangle c_{\sigma} - \langle \sigma|\hat{J}_1(\mathbf{r})|0 \rangle \bar{c}_{\sigma}]}{\sum_{\sigma}[\langle 0|\hat{J}_3(\mathbf{r})|\sigma \rangle c_{\sigma} - \langle \sigma|\hat{J}_3(\mathbf{r})|0 \rangle \bar{c}_{\sigma}]}.$$

Remembering the definition of ξ_i and c_{σ} , multiplying the numerator and the denominator of the right hand side by $(\Omega - \Omega_\nu)$ and taking the limit $\Omega \rightarrow \Omega_\nu$ we arrive to the differential equation

$$\frac{dx}{dz} = \frac{\langle 0|\hat{J}_1(\mathbf{r})|\nu \rangle}{\langle 0|\hat{J}_3(\mathbf{r})|\nu \rangle}, \quad (111)$$

which determines the lines of currents in the plane $y = const$ for the resonance state $|\nu \rangle$.

6.3 Green's function method

The distribution function being known, one can calculate the distribution of nuclear currents $\mathbf{j}^\tau(\mathbf{r}, t) = m\rho^\tau(\mathbf{r}, t)\mathbf{u}^\tau(\mathbf{r}, t)$. There are no any currents in the equilibrium state, so we have

$$j_x^\tau(\mathbf{r}, t) = m\rho^\tau(\mathbf{r}, t)\delta u_x^\tau(\mathbf{r}, t) = \int \frac{2d^3p}{(2\pi\hbar)^3} p_x f_1^\tau(\mathbf{r}, \mathbf{p}, t)$$

$$\begin{aligned}
&= -\frac{z}{2m\omega_x}[C_+^\tau(t)\omega_+ + C_-^\tau(t)\omega_-] \int \frac{2d^3p}{(2\pi\hbar)^3} p_x^2 \delta(\epsilon_F - H_0) \\
&= -\frac{z}{\omega_x} \frac{2\pi}{3} [2m\epsilon_F - m^2\omega_x^2(x^2 + y^2) - m^2\omega_z^2 z^2]^{3/2} [C_+^\tau(t)\omega_+ + C_-^\tau(t)\omega_-] \\
&= -\frac{z}{2m\omega_x} \rho_0^\tau(\mathbf{r}) [C_+^\tau(t)\omega_+ + C_-^\tau(t)\omega_-],
\end{aligned} \tag{112}$$

where the following notation is introduced

$$C_\pm^\tau(t) = \int_{-\infty}^t dt' Z_1^\tau(t') \cos \omega_\pm(t' - t).$$

Deriving (112) we used the approximation (11) which means, in particular, that $\omega_i^n = \omega_i^p$ and $\rho_0^n = \rho_0^p = \rho_0/2$. Another component of the flow is

$$j_z^\tau(\mathbf{r}, t) = \int \frac{2d^3p}{(2\pi\hbar)^3} p_z f_1^\tau(\mathbf{r}, \mathbf{p}, t) = -\frac{x+y}{2m\omega_z} \rho_0^\tau(\mathbf{r}) [C_+^\tau(t)\omega_+ - C_-^\tau(t)\omega_-], \tag{113}$$

With the help of the isovector counterpart of formulae (91) the functions $C_\pm^\tau(t)$ can be written via dynamical variables

$$C_-^\tau(t) = [L_1^\tau(t) + \frac{\omega_+}{\omega_-} I_y^\tau(t)]/\zeta, \quad C_+^\tau(t) = -[L_1^\tau(t) + \frac{\omega_-}{\omega_+} I_y^\tau(t)]/\zeta$$

with $\zeta = \frac{2\pi^3\epsilon_F^4}{3m\omega_x^3\omega_z^2} \frac{2}{(2\pi\hbar)^3} = \beta m\omega_x\omega_z/6\kappa_0$ and the required combinations are

$$\begin{aligned}
C_+^\tau(t)\omega_+ + C_-^\tau(t)\omega_- &= -2\omega_z[L_1^\tau(t) - I_y^\tau(t)]/\zeta, \\
C_+^\tau(t)\omega_+ - C_-^\tau(t)\omega_- &= -2\omega_x[L_1^\tau(t) + I_y^\tau(t)]/\zeta
\end{aligned} \tag{114}$$

We are interested in isovector flows $\bar{j}_x = j_x^n - j_x^p$ and $\bar{j}_z = j_z^n - j_z^p$. With the help of the first and last equations of (17) we find

$$\begin{aligned}
\bar{C}_+^\tau(t)\omega_+ + \bar{C}_-^\tau(t)\omega_- &= -2\omega_z i\Omega m [1 + 2\frac{\bar{\omega}^2}{\Omega^2}(1 - \alpha)\delta] \bar{Q}_1/\zeta, \\
\bar{C}_+^\tau(t)\omega_+ - \bar{C}_-^\tau(t)\omega_- &= -2\omega_x i\Omega m [1 - 2\frac{\bar{\omega}^2}{\Omega^2}(1 - \alpha)\delta] \bar{Q}_1/\zeta.
\end{aligned} \tag{115}$$

As a result, we have the explicit expressions for the currents

$$\begin{aligned}
\bar{j}_z(\mathbf{r}, t) &= \frac{i\Omega}{2\zeta} [1 - 2\frac{\bar{\omega}^2}{\Omega^2}(1 - \alpha)\delta] \bar{Q}_1(t) \rho_0(\mathbf{r}) \frac{\omega_x}{\omega_z} (x + y), \\
\bar{j}_x(\mathbf{r}, t) &= \frac{i\Omega}{2\zeta} [1 + 2\frac{\bar{\omega}^2}{\Omega^2}(1 - \alpha)\delta] \bar{Q}_1(t) \rho_0(\mathbf{r}) \frac{\omega_z}{\omega_x} z.
\end{aligned} \tag{116}$$

Following the recipe of section 6.1 we can derive the differential equation for lines of currents, for example, in the plane $y = 0$:

$$\frac{dx}{dz} = \frac{\bar{j}_x}{\bar{j}_z} \longrightarrow \frac{dx}{dz} = \frac{z \omega_z^2}{x \omega_x^2} \frac{1 + 2 \frac{\bar{\omega}^2}{\Omega^2} (1 - \alpha) \delta}{1 - 2 \frac{\bar{\omega}^2}{\Omega^2} (1 - \alpha) \delta} = \frac{z}{x} \frac{B}{A} \quad (117)$$

with A and B defined by (104). Obviously, this expression coincides exactly with formula (105). It is necessary to emphasize the principal point: the result (117) is obtained from the GF method in a direct way, whereas deriving formula (105) we made the strong approximation about truncating the expansion (102) which parametrizes the displacements. The agreement of both expressions tells us about the internal consistency of the various approaches to obtain the gross structure of the flow patterns.

6.4 Summary of flow calculations

In conclusion in full RPA one must calculate the currents numerically leading to fine details (shell effects) whereas in WFM and GF treatments one obtains their gross structure with analytical formulas. The latter feature is quite important in order to understand the real character of the motion under study since current patterns produced numerically from complicated formulas with a lot of summations like in (110) can hardly be interpreted physically. A good example is the interplay of the scissors mode and the isovector giant quadrupole resonance. Looking only at the flow patterns (see Figs. 1, 2) one would not be able to tell that the former is mostly rotational with a small amount of an irrotational component and the other way round for the latter, as this can be seen from eqs. (106, 107).

7 Conclusion

In this paper we made an exhaustive comparison of different methods to treat collective excitations in nuclei, like the scissors mode, isovector and isoscalar giant quadrupole resonances. This comparison was exemplified on the H.O. plus separable quadrupole–quadrupole force model but it has more general character.

We investigated WFM, RPA and Green’s Function (GF) methods. Under certain circumstances all three methods give essentially the same results. For example all methods give in

our model the same analytical expressions for energies and transition probabilities for all the excitations considered. It turned out that WFM and GF methods are very close to one another. Contrary to RPA, both work in phase space and incorporate semiclassical aspects, with no need to introduce a single particle basis. Finally both methods yield identical sets of dynamical equations for the moments. However, in the case of realistic forces the GF method loses its simplicity and the more complicated pseudo particle method has to be applied, whereas the WFM method continues to be a convenient and powerful tool for the description of the collective motions as it was demonstrated in ref. [6, 25, 26] employing Skyrme forces.

To show the analytical equivalence between WFM and RPA methods one needs to introduce the dynamical equations for the transition matrix elements. They can be derived either from the RPA equations for the amplitudes X_{kq}, Y_{kq} or from the WFM dynamical equations for the moments. This proves the identity of eigenvalues in both methods under the condition that a complete basis is used in both cases. However, both methods behave differently when the dimension of the space is reduced. Actually WFM is designed to use only rather few moments of low rank. The restricted number of eigenvalues approximate the collective states in an optimal way, representing e.g. their centroid positions, as this was shown in [6, 25, 26, 27]. In this sense WFM has similarity with the sum rule approach [4] which works, however, only in the cases when practically all strength is exhausted by one state, whereas WFM method works also in situations when the strength is distributed among several excitations. On the contrary in RPA one needs in general a rather large space to correctly account for the collectivity of e.g. the giant resonances. At the same time a certain fine structure of the resonances is also obtained. Concerning the spectrum both methods are thus complementary. The situation is different for the currents and flow patterns. Since RPA is a fully quantal approach, the current lines can even in our simplified model be calculated only numerically. They show a complicated pattern due to the shell effects. Without further efforts one will not be able to analyze the nature of the flows. A good example is given in our model with the low and high lying scissors mode (the latter being the IVGQR). Due to the analytic formulas found with WFM and GF methods which naturally lead to smooth current distributions free of rapidly fluctuating behavior from shell effects, we were able to show that the low lying scissors mode is mostly rotational with a

slight admixture of an irrotational component and the other way round for the IVGQR.

In addition to our earlier work we investigated in detail the so called synthetic scissors mode which is based on the picture of two counter rotating proton and neutron mass distributions. Calculating the overlap of this synthetic scissors mode with the real one we could show that the squared overlap amounts only to about 60% in the best of all cases. We also showed explicitly the orthogonality of the spurious mode to all other “intrinsic” excitations of the model.

Future work in this direction shall be concerned with the scissors mode in neutron rich nuclei and with the consideration of superfluidity.

Appendix A

It is known that the deformed harmonic oscillator Hamiltonian can be obtained in a Hartree approximation “by making the assumption that the isoscalar part of the QQ force builds the one-body container well” [10]. In our case it is obtained quite easily by summing the expressions for V^p and V^n (formula (7)):

$$V(\mathbf{r}, t) = \frac{1}{2}(V^p(\mathbf{r}, t) + V^n(\mathbf{r}, t)) = \frac{1}{2}m\omega^2 r^2 + \kappa_0 \sum_{\mu=-2}^2 (-1)^\mu Q_{2-\mu}(t) q_{2\mu}(\mathbf{r}). \quad (\text{A.1})$$

In the state of equilibrium (i.e., in the absence of an external field) $Q_{2\pm 1} = Q_{2\pm 2} = 0$. Using the definition [31] $Q_{20} = Q_{00}\frac{4}{3}\delta$ and the formula $q_{20} = 2z^2 - x^2 - y^2$ we obtain the potential of the anisotropic harmonic oscillator

$$V(\mathbf{r}) = \frac{m}{2}[\omega_x^2(x^2 + y^2) + \omega_z^2 z^2]$$

with oscillator frequencies

$$\omega_x^2 = \omega_y^2 = \omega^2(1 + \sigma\delta), \quad \omega_z^2 = \omega^2(1 - 2\sigma\delta),$$

where $\sigma = -\kappa_0 \frac{8Q_{00}}{3m\omega^2}$. The definition of the deformation parameter δ must be reproduced by the harmonic oscillator wave functions, which allows one to fix the value of σ . We have

$$Q_{00} = \frac{\hbar}{m} \left(\frac{\Sigma_x}{\omega_x} + \frac{\Sigma_y}{\omega_y} + \frac{\Sigma_z}{\omega_z} \right), \quad Q_{20} = 2 \frac{\hbar}{m} \left(\frac{\Sigma_z}{\omega_z} - \frac{\Sigma_x}{\omega_x} \right),$$

where $\Sigma_x = \Sigma_{i=1}^A (n_x + \frac{1}{2})_i$ and n_x is the oscillator quantum number. Using the self-consistency condition [31]

$$\Sigma_x \omega_x = \Sigma_y \omega_y = \Sigma_z \omega_z = \Sigma_0 \omega_0,$$

where Σ_0 and ω_0 are defined in the spherical case, we get

$$\frac{Q_{20}}{Q_{00}} = 2 \frac{\omega_x^2 - \omega_z^2}{\omega_x^2 + 2\omega_z^2} = \frac{2\sigma\delta}{1 - \sigma\delta} = \frac{4}{3}\delta.$$

Solving the last equation with respect to σ , we find

$$\sigma = \frac{2}{3 + 2\delta}. \quad (\text{A.2})$$

Therefore, the oscillator frequencies and the strength constant can be written as

$$\omega_x^2 = \omega_y^2 = \bar{\omega}^2(1 + \frac{4}{3}\delta), \quad \omega_z^2 = \bar{\omega}^2(1 - \frac{2}{3}\delta), \quad \kappa_0 = -\frac{m\bar{\omega}^2}{4Q_{00}} \quad (\text{A.3})$$

with $\bar{\omega}^2 = \omega^2/(1 + \frac{2}{3}\delta)$. The condition for volume conservation $\omega_x \omega_y \omega_z = \text{const} = \omega_0^3$ makes ω δ -dependent

$$\omega^2 = \omega_0^2 \frac{1 + \frac{2}{3}\delta}{(1 + \frac{4}{3}\delta)^{2/3}(1 - \frac{2}{3}\delta)^{1/3}}.$$

So the final expressions for oscillator frequencies are

$$\omega_x^2 = \omega_y^2 = \omega_0^2 \left(\frac{1 + \frac{4}{3}\delta}{1 - \frac{2}{3}\delta} \right)^{1/3}, \quad \omega_z^2 = \omega_0^2 \left(\frac{1 - \frac{2}{3}\delta}{1 + \frac{4}{3}\delta} \right)^{2/3}. \quad (\text{A.4})$$

It is interesting to compare these expressions with the very popular [31, 4] parametrization

$$\omega_x^2 = \omega_y^2 = \omega'^2(1 + \frac{2}{3}\delta'), \quad \omega_z^2 = \omega'^2(1 - \frac{4}{3}\delta').$$

The volume conservation condition gives

$$\omega'^2 = \frac{\omega_0^2}{(1 + \frac{2}{3}\delta')^{2/3}(1 - \frac{4}{3}\delta')^{1/3}},$$

so the final expressions for oscillator frequencies are

$$\omega_x^2 = \omega_y^2 = \omega_0^2 \left(\frac{1 + \frac{2}{3}\delta'}{1 - \frac{4}{3}\delta'} \right)^{1/3}, \quad \omega_z^2 = \omega_0^2 \left(\frac{1 - \frac{4}{3}\delta'}{1 + \frac{2}{3}\delta'} \right)^{2/3}. \quad (\text{A.5})$$

The direct comparison of expressions (A.4) and (A.5) allows one to establish the following relation between δ and δ' :

$$\delta' = \frac{\delta}{1 + 2\delta}, \quad \delta = \frac{\delta'}{1 - 2\delta'}.$$

One more parametrization of oscillator frequencies can be found in the review [21]:

$$\omega_x^2 = \omega_y^2 = \frac{\omega''^2}{1 - \frac{2}{3}\delta''}, \quad \omega_z^2 = \frac{\omega''^2}{1 + \frac{4}{3}\delta''}.$$

One has from the volume conservation condition

$$\omega''^2 = \omega_0^2(1 - \frac{2}{3}\delta'')^{2/3}(1 + \frac{4}{3}\delta'')^{1/3},$$

so the final expressions for oscillator frequencies are

$$\omega_x^2 = \omega_y^2 = \omega_0^2 \left(\frac{1 + \frac{4}{3}\delta''}{1 - \frac{2}{3}\delta''} \right)^{1/3}, \quad \omega_z^2 = \omega_0^2 \left(\frac{1 - \frac{2}{3}\delta''}{1 + \frac{4}{3}\delta''} \right)^{2/3}, \quad (\text{A.6})$$

that coincide exactly with (A.4), i.e. $\delta'' = \delta$.

It is easy to see that equations (A.4) correspond to the case when the deformed density $\rho(\mathbf{r})$ is obtained from the spherical density $\rho_0(r)$ by the scale transformation [9]

$$(x, y, z) \rightarrow (xe^{\alpha/2}, ye^{\alpha/2}, ze^{-\alpha})$$

with

$$e^\alpha = \left(\frac{1 + \frac{4}{3}\delta}{1 - \frac{2}{3}\delta} \right)^{1/3}, \quad \delta = \frac{3e^{3\alpha} - 1}{2e^{3\alpha} + 2}, \quad (\text{A.7})$$

which conserves the volume and does not destroy the self-consistency, because the density and potential are transformed in the same way.

It is necessary to note that Q_{00} also depends on δ

$$Q_{00} = \frac{\hbar}{m} \left(\frac{\Sigma_x}{\omega_x} + \frac{\Sigma_y}{\omega_y} + \frac{\Sigma_z}{\omega_z} \right) = \frac{\hbar}{m} \Sigma_0 \omega_0 \left(\frac{2}{\omega_x^2} + \frac{1}{\omega_z^2} \right) = Q_{00}^0 \frac{1}{(1 + \frac{4}{3}\delta)^{1/3}(1 - \frac{2}{3}\delta)^{2/3}},$$

where $Q_{00}^0 = A^{\frac{3}{5}} R^2$, $R = r_0 A^{1/3}$. As a result, the final expression for the strength constant becomes

$$\kappa_0 = -\frac{m\omega_0^2}{4Q_{00}^0} \left(\frac{1 - \frac{2}{3}\delta}{1 + \frac{4}{3}\delta} \right)^{1/3} = -\frac{m\omega_0^2}{4Q_{00}^0} e^{-\alpha},$$

that coincides with the respective result of [9].

Appendix B

To calculate the sums $\mathcal{Q}_0 = \sum_{mi(\Delta N=0)} |\mathcal{Q}_{mi}|^2$ and $\mathcal{Q}_2 = \sum_{mi(\Delta N=2)} |\mathcal{Q}_{mi}|^2$ we employ the sum-rule techniques of Suzuki and Rowe [9]. The well known harmonic oscillator relations

$$x\psi_{n_x} = \sqrt{\frac{\hbar}{2m\omega_x}} (\sqrt{n_x}\psi_{n_x-1} + \sqrt{n_x+1}\psi_{n_x+1}),$$

$$\hat{p}_x \psi_{n_x} = -i \sqrt{\frac{m\hbar\omega_x}{2}} (\sqrt{n_x} \psi_{n_x-1} - \sqrt{n_x+1} \psi_{n_x+1}) \quad (\text{B.1})$$

allow us to write

$$\begin{aligned} xz \psi_{n_x} \psi_{n_z} &= \frac{\hbar}{2m\sqrt{\omega_x\omega_z}} (\sqrt{n_x n_z} \psi_{n_x-1} \psi_{n_z-1} + \sqrt{(n_x+1)(n_z+1)} \psi_{n_x+1} \psi_{n_z+1} \\ &\quad + \sqrt{(n_x+1)n_z} \psi_{n_x+1} \psi_{n_z-1} + \sqrt{n_x(n_z+1)} \psi_{n_x-1} \psi_{n_z+1}), \\ \frac{\hat{p}_x \hat{p}_z}{m^2 \omega_x \omega_z} \psi_{n_x} \psi_{n_z} &= -\frac{\hbar}{2m\sqrt{\omega_x\omega_z}} (\sqrt{n_x n_z} \psi_{n_x-1} \psi_{n_z-1} + \sqrt{(n_x+1)(n_z+1)} \psi_{n_x+1} \psi_{n_z+1} \\ &\quad - \sqrt{(n_x+1)n_z} \psi_{n_x+1} \psi_{n_z-1} - \sqrt{n_x(n_z+1)} \psi_{n_x-1} \psi_{n_z+1}). \end{aligned} \quad (\text{B.2})$$

These formulae demonstrate in an obvious way that the operators

$$P_0 = \frac{1}{2} (zx + \frac{1}{m^2 \omega_x \omega_z} \hat{p}_x \hat{p}_z) \quad \text{and} \quad P_2 = \frac{1}{2} (zx - \frac{1}{m^2 \omega_x \omega_z} \hat{p}_x \hat{p}_z)$$

contribute only to the excitation of the $\Delta N = 0$ and $\Delta N = 2$ states, respectively. Following [9], we express the zx component of $r^2 Y_{21} = \sqrt{\frac{5}{16\pi}} \mathcal{Q} = -\sqrt{\frac{15}{8\pi}} z(x + iy)$ as

$$zx = P_0 + P_2.$$

Hence, we have

$$\begin{aligned} \epsilon_0 \sum_{mi(\Delta N=0)} | \langle 0 | \sum_{s=1}^A z_s x_s | mi \rangle |^2 &= \epsilon_0 \sum_{mi} | \langle 0 | \sum_{s=1}^A P_0(s) | mi \rangle |^2 \\ &= \frac{1}{2} \langle 0 | [\sum_{s=1}^A P_0(s), [H, \sum_{s=1}^A P_0(s)]] | 0 \rangle, \end{aligned} \quad (\text{B.3})$$

where $\epsilon_0 = \hbar(\omega_x - \omega_z)$. The above commutator is easily evaluated for the Hamiltonian with the potential (A.1), as

$$\langle 0 | [\sum_{s=1}^A P_0(s), [H, \sum_{s=1}^A P_0(s)]] | 0 \rangle = \frac{\hbar}{2m} \epsilon_0 \left(\frac{\langle 0 | \sum_{s=1}^A z_s^2 | 0 \rangle}{\omega_x} - \frac{\langle 0 | \sum_{s=1}^A x_s^2 | 0 \rangle}{\omega_z} \right). \quad (\text{B.4})$$

Taking into account the axial symmetry and using the definitions

$$Q_{00} = \langle 0 | \sum_{s=1}^A (2x_s^2 + z_s^2) | 0 \rangle, \quad Q_{20} = 2 \langle 0 | \sum_{s=1}^A (z_s^2 - x_s^2) | 0 \rangle, \quad Q_{20} = Q_{00} \frac{4}{3} \delta,$$

we transform this expression to

$$\langle 0 | [\sum_{s=1}^A P_0(s), [H, \sum_{s=1}^A P_0(s)]] | 0 \rangle = \frac{\hbar}{6m} \epsilon_0 Q_{00} \left(\frac{1 + \frac{4}{3}\delta}{\omega_x} - \frac{1 - \frac{2}{3}\delta}{\omega_z} \right). \quad (\text{B.5})$$

With the help of the self-consistent expressions for ω_x, ω_z (A.3) one comes to the following result:

$$\langle 0 | [\sum_{s=1}^A P_0(s), [H, \sum_{s=1}^A P_0(s)]] | 0 \rangle = \frac{Q_{00}}{6m} \frac{\epsilon_0^2}{\bar{\omega}^2} = \frac{\hbar^2}{6m} Q_{00}^0 \left(\frac{\omega_0}{\omega_z} - \frac{\omega_0}{\omega_x} \right)^2. \quad (\text{B.6})$$

By using the fact that the matrix elements for the zy component of $r^2 Y_{21}$ are identical to those for the zx component, because of axial symmetry, we finally obtain

$$\epsilon_0 \sum_{mi(\Delta N=0)} | \langle 0 | \sum_{s=1}^A r_s^2 Y_{21} | mi \rangle |^2 = \frac{5}{16\pi} \frac{Q_{00}}{m \bar{\omega}^2} \epsilon_0^2 = \frac{5}{16\pi} \frac{Q_{00}^0}{m} \frac{\epsilon_0^2}{\omega_0^2} \left(\frac{1 + \frac{4}{3}\delta}{1 - \frac{2}{3}\delta} \right)^{1/3}. \quad (\text{B.7})$$

By calculating the double commutator for the P_2 operator, we find

$$\epsilon_2 \sum_{mi(\Delta N=2)} | \langle 0 | \sum_{s=1}^A r_s^2 Y_{21} | mi \rangle |^2 = \frac{5}{16\pi} \frac{Q_{00}}{m \bar{\omega}^2} \epsilon_2^2 = \frac{5}{16\pi} \frac{Q_{00}^0}{m} \frac{\epsilon_2^2}{\omega_0^2} \left(\frac{1 + \frac{4}{3}\delta}{1 - \frac{2}{3}\delta} \right)^{1/3}, \quad (\text{B.8})$$

where $\epsilon_2 = \hbar(\omega_x + \omega_z)$.

We need also the sums Q_0^τ and Q_2^τ calculated separately for neutron and proton systems with the mean fields V^n and V^p , respectively. The necessary formulae are easily derivable from the already obtained results. There are no reasons to require the fulfillment of the self-consistency conditions for neutrons and protons separately, so one has to use formula (B.5). A trivial change of notation gives

$$\langle 0 | [\sum_{s=1}^Z P_0(s), [H^p, \sum_{s=1}^Z P_0(s)]] | 0 \rangle = \frac{\hbar}{6m} \epsilon_0^p Q_{00}^p \left(\frac{1 + \frac{4}{3}\delta^p}{\omega_x^p} - \frac{1 - \frac{2}{3}\delta^p}{\omega_z^p} \right), \quad (\text{B.9})$$

$$\epsilon_0^p \sum_{mi(\Delta N=0)} | \langle 0 | \sum_{s=1}^Z r_s^2 Y_{21} | mi \rangle |^2 = \frac{5}{16\pi} \frac{\hbar}{m} \epsilon_0^p Q_{00}^p \left(\frac{1 + \frac{4}{3}\delta^p}{\omega_x^p} - \frac{1 - \frac{2}{3}\delta^p}{\omega_z^p} \right), \quad (\text{B.10})$$

$$\epsilon_2^p \sum_{mi(\Delta N=2)} | \langle 0 | \sum_{s=1}^Z r_s^2 Y_{21} | mi \rangle |^2 = \frac{5}{16\pi} \frac{\hbar}{m} \epsilon_2^p Q_{00}^p \left(\frac{1 + \frac{4}{3}\delta^p}{\omega_x^p} + \frac{1 - \frac{2}{3}\delta^p}{\omega_z^p} \right). \quad (\text{B.11})$$

The nontrivial information is contained in oscillator frequencies of the mean fields V^p and V^n (formula (7))

$$\begin{aligned} (\omega_x^p)^2 &= \omega^2 \left[1 - \frac{2}{m\omega^2} (\kappa Q_{20}^p + \bar{\kappa} Q_{20}^n) \right], & (\omega_z^p)^2 &= \omega^2 \left[1 + \frac{4}{m\omega^2} (\kappa Q_{20}^p + \bar{\kappa} Q_{20}^n) \right], \\ (\omega_x^n)^2 &= \omega^2 \left[1 - \frac{2}{m\omega^2} (\kappa Q_{20}^n + \bar{\kappa} Q_{20}^p) \right], & (\omega_z^n)^2 &= \omega^2 \left[1 + \frac{4}{m\omega^2} (\kappa Q_{20}^n + \bar{\kappa} Q_{20}^p) \right]. \end{aligned} \quad (\text{B.12})$$

The above-written formulae can also be used to calculate the analogous sums for operators containing various combinations of momenta and coordinates, for example, components of an angular momentum, tensor products $(r\hat{p})_{21}$ and $(\hat{p}^2)_{21}$. By definition $\hat{I}_1 = y\hat{p}_z - z\hat{p}_y$, $\hat{I}_2 = z\hat{p}_x - x\hat{p}_z$. In accordance with (B.1), we have

$$\begin{aligned} x\hat{p}_z\psi_{n_x}\psi_{n_z} = & -i\frac{\hbar}{2}\sqrt{\frac{\omega_z}{\omega_x}}(\sqrt{n_x n_z}\psi_{n_x-1}\psi_{n_z-1} - \sqrt{(n_x+1)(n_z+1)}\psi_{n_x+1}\psi_{n_z+1}) \\ & + \sqrt{(n_x+1)n_z}\psi_{n_x+1}\psi_{n_z-1} - \sqrt{n_x(n_z+1)}\psi_{n_x-1}\psi_{n_z+1}. \end{aligned} \quad (\text{B.13})$$

Therefore,

$$\begin{aligned} \hat{I}_2\psi_{n_x}\psi_{n_z} = & i\frac{\hbar}{2}\left(\sqrt{\frac{\omega_z}{\omega_x}} - \sqrt{\frac{\omega_x}{\omega_z}}\right)(\sqrt{n_x n_z}\psi_{n_x-1}\psi_{n_z-1} - \sqrt{(n_x+1)(n_z+1)}\psi_{n_x+1}\psi_{n_z+1}) \\ & + i\frac{\hbar}{2}\left(\sqrt{\frac{\omega_z}{\omega_x}} + \sqrt{\frac{\omega_x}{\omega_z}}\right)(\sqrt{(n_x+1)n_z}\psi_{n_x+1}\psi_{n_z-1} - \sqrt{n_x(n_z+1)}\psi_{n_x-1}\psi_{n_z+1}). \end{aligned} \quad (\text{B.14})$$

Having formulae (B.2) and (B.14), one derives the following expressions for matrix elements coupling the ground state with $\Delta N = 2$ and $\Delta N = 0$ excitations:

$$\begin{aligned} \langle n_x+1, n_z+1 | \hat{I}_2 | 0 \rangle &= i\frac{\hbar}{2}\frac{(\omega_x^2 - \omega_z^2)}{\omega_x + \omega_z}\sqrt{\frac{(n_x+1)(n_z+1)}{\omega_x\omega_z}}, \\ \langle n_x+1, n_z-1 | \hat{I}_2 | 0 \rangle &= i\frac{\hbar}{2}\frac{(\omega_x^2 - \omega_z^2)}{\omega_x - \omega_z}\sqrt{\frac{(n_x+1)n_z}{\omega_x\omega_z}}, \\ \langle n_x+1, n_z+1 | xz | 0 \rangle &= \frac{\hbar}{2m}\sqrt{\frac{(n_x+1)(n_z+1)}{\omega_x\omega_z}}, \\ \langle n_x+1, n_z-1 | xz | 0 \rangle &= \frac{\hbar}{2m}\sqrt{\frac{(n_x+1)n_z}{\omega_x\omega_z}}. \end{aligned} \quad (\text{B.15})$$

It is easy to see that

$$\begin{aligned} \langle n_x+1, n_z+1 | \hat{I}_2 | 0 \rangle &= im\frac{(\omega_x^2 - \omega_z^2)}{\omega_x + \omega_z} \langle n_x+1, n_z+1 | xz | 0 \rangle, \\ \langle n_x+1, n_z-1 | \hat{I}_2 | 0 \rangle &= im\frac{(\omega_x^2 - \omega_z^2)}{\omega_x - \omega_z} \langle n_x+1, n_z-1 | xz | 0 \rangle. \end{aligned}$$

Due to the degeneracy of the model all particle hole excitations with $\Delta N = 2$ have the same energy ϵ_2 and all particle hole excitations with $\Delta N = 0$ have the energy ϵ_0 . This fact allows one to join the last two formulae into one general expression

$$\langle ph | \hat{I}_2 | 0 \rangle = i\hbar m \frac{(\omega_x^2 - \omega_z^2)}{\epsilon_{ph}} \langle ph | xz | 0 \rangle.$$

Taking into account the axial symmetry we have an analogous formula for \hat{I}_1 :

$$\langle ph|\hat{I}_1|0 \rangle = -i\hbar m \frac{(\omega_x^2 - \omega_z^2)}{\epsilon_{ph}} \langle ph|yz|0 \rangle .$$

The magnetic transition operator (29) is proportional to the angular momentum: $\hat{f}_{1\pm 1} = -\frac{ie}{4mc}\sqrt{\frac{3}{2\pi}}(\hat{I}_2 \mp i\hat{I}_1)$ Therefore, we can write

$$\langle ph|\hat{f}_{1\pm 1}|0 \rangle = -\frac{e\hbar}{2c\sqrt{5}} \frac{(\omega_x^2 - \omega_z^2)}{\epsilon_{ph}} \langle ph|r^2 Y_{2\pm 1}|0 \rangle . \quad (\text{B.16})$$

Similar calculations for the tensor product $(r\hat{p})_{21} = -\frac{1}{2}[z\hat{p}_x + x\hat{p}_z + i(z\hat{p}_y + y\hat{p}_z)]$ lead to the following relation:

$$\langle ph|(r\hat{p})_{21}|0 \rangle = i\frac{m}{\hbar}\sqrt{\frac{2\pi}{15}}\epsilon_{ph} \langle ph|r^2 Y_{2\pm 1}|0 \rangle = i\frac{m}{2\hbar}\epsilon_{ph} \langle ph|r_{21}^2|0 \rangle . \quad (\text{B.17})$$

Two kinds of particle hole matrix elements are obtained from the second formula of (B.2):

$$\begin{aligned} \langle n_x + 1, n_z + 1|\hat{p}_x\hat{p}_z|0 \rangle &= -\hbar m\omega_x\omega_z \sqrt{\frac{(n_x + 1)(n_z + 1)}{2\omega_x 2\omega_z}}, \\ \langle n_x + 1, n_z - 1|\hat{p}_x\hat{p}_z|0 \rangle &= \hbar m\omega_x\omega_z \sqrt{\frac{(n_x + 1)n_z}{2\omega_x 2\omega_z}}. \end{aligned}$$

Simple comparison with (B.15) shows that

$$\begin{aligned} \langle n_x + 1, n_z + 1|\hat{p}_x\hat{p}_z|0 \rangle &= -m^2\omega_x\omega_z \langle n_x + 1, n_z + 1|xz|0 \rangle, \\ \langle n_x + 1, n_z - 1|\hat{p}_x\hat{p}_z|0 \rangle &= m^2\omega_x\omega_z \langle n_x + 1, n_z - 1|xz|0 \rangle . \end{aligned}$$

With the help of the obvious relations

$$2\omega_x\omega_z = \omega_x^2 + \omega_z^2 - \epsilon_0^2/\hbar^2, \quad -2\omega_x\omega_z = \omega_x^2 + \omega_z^2 - \epsilon_z^2/\hbar^2$$

these two formulae can be joined into one expression

$$\langle ph|\hat{p}_x\hat{p}_z|0 \rangle = \frac{m^2}{2}(\omega_x^2 + \omega_z^2 - \epsilon_{ph}^2/\hbar^2) \langle ph|xz|0 \rangle .$$

By definition $\hat{p}_{21}^2 = -\hat{p}_z(\hat{p}_x + i\hat{p}_y)$ and $\hat{r}_{21}^2 = -z(x + iy)$, hence,

$$\langle ph|\hat{p}_{21}^2|0 \rangle = \frac{m^2}{2}(\omega_x^2 + \omega_z^2 - \epsilon_{ph}^2/\hbar^2) \langle ph|r_{21}^2|0 \rangle . \quad (\text{B.18})$$

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